烟台理工学院

Yantai Institute of Technology

（原烟台大学文经学院）

(Wenjing College Yantai University)

机器人学

人工智能学院　杨智勇

二零二一年八月二十日

2021-9-22

第二章 空间描述和变换

2.1 导读

- 一個剛體(Rigid body)的狀態該如何描述？
  - 平面：

$\{W\}$ world frame

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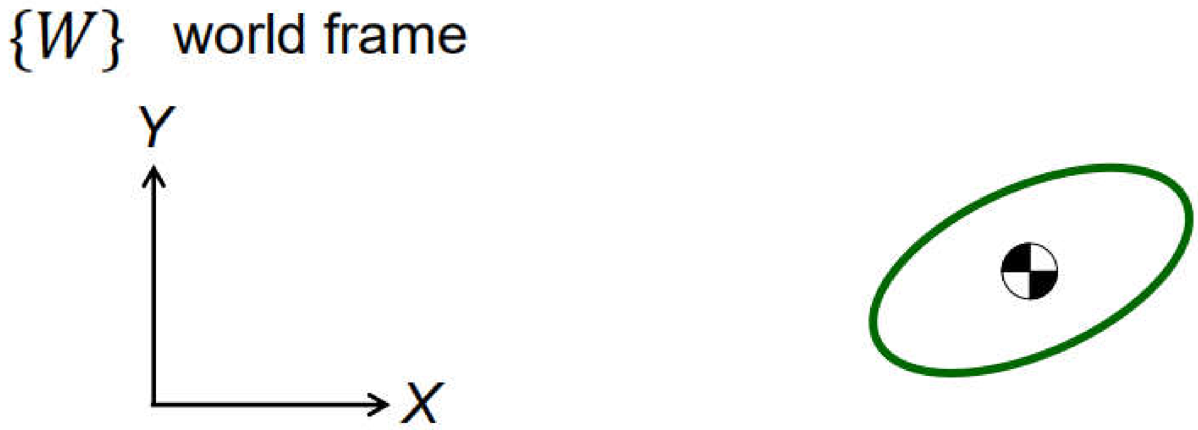
  - 平面：

    移動 2 DOFs、轉動 1 DOF Degree of freedom



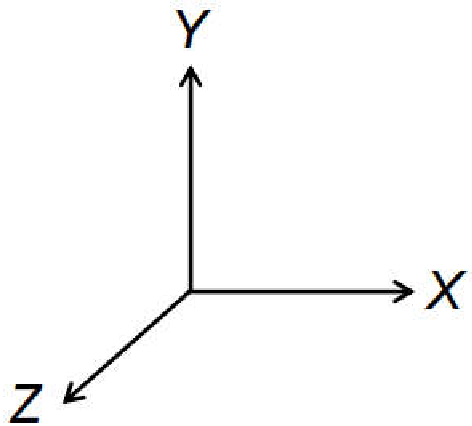
2.1 导读

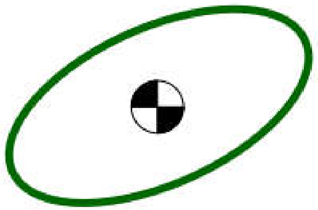
- 一個剛體(Rigid body)的狀態該如何描述？
  - 平面：

    移動 2 DOFs、轉動 1 DOF Degree of freedom
  - 空間：

    移動 3 DOFs、轉動 3 DOFs

$\{W\}$ world frame

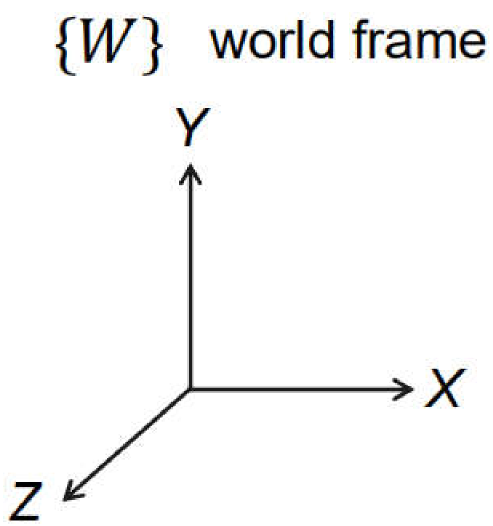


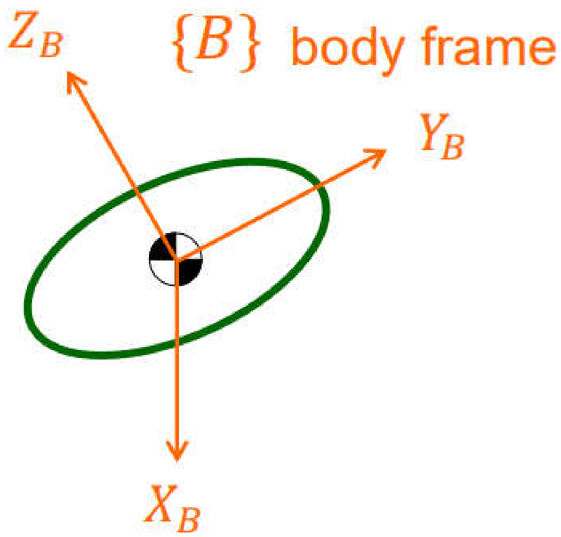


2.1 导读

- 該如何整合表達剛體的狀態？

  ⇨ 在剛體(Rigid body)上建立frame，常建立在質心上



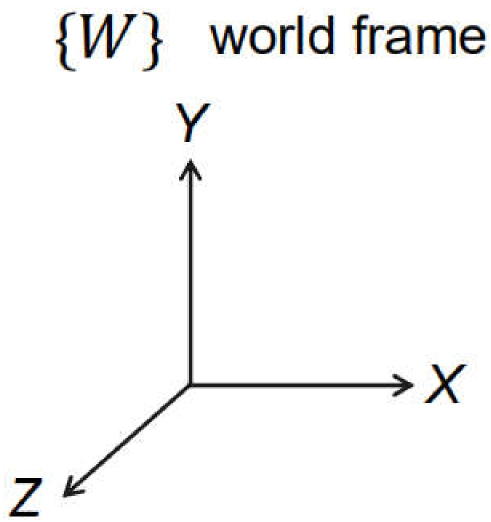


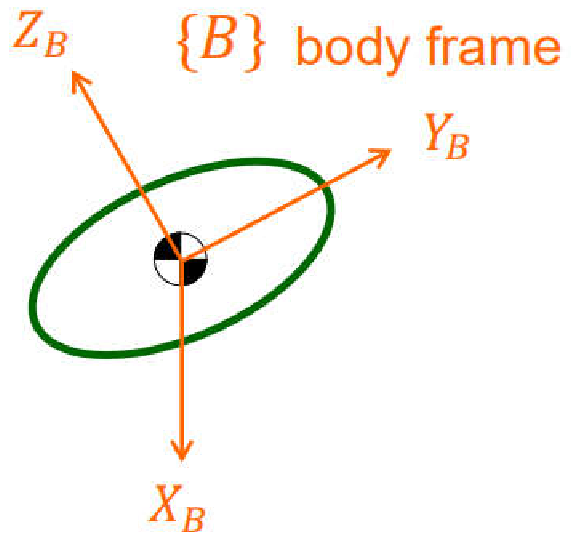
2.1 导读

- 該如何整合表達剛體的狀態？

⇨ 在剛體(Rigid body)上建立frame，常建立在質心上

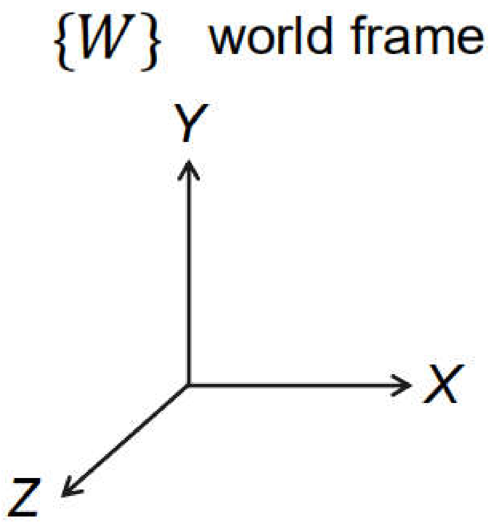
- 移動：由body frame 的原點位置判定
- 轉動：由body frame的姿態判定

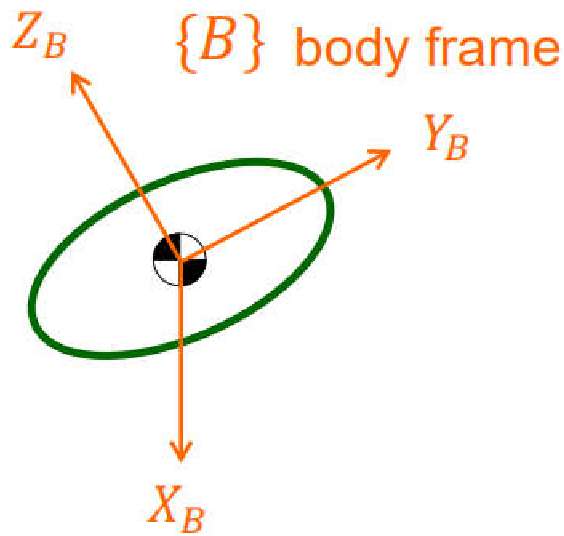




2.1 导读

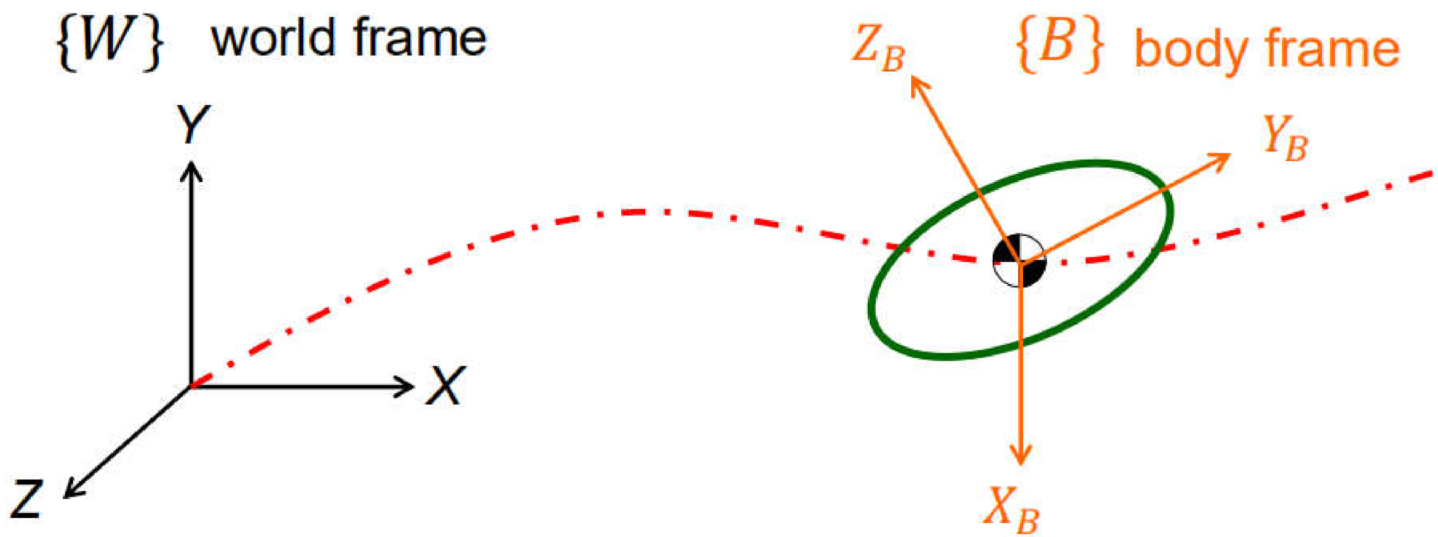
- 一個剛體(Rigid body)的「運動」狀態該如何描述？





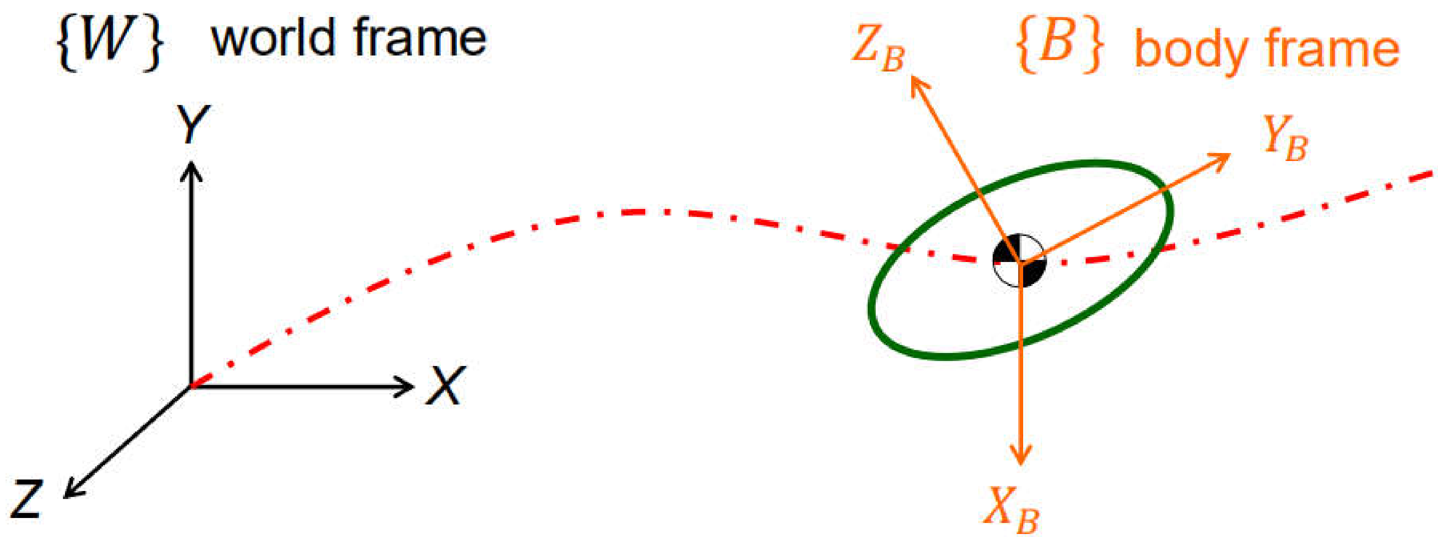
2.1 导读

- 一個剛體(Rigid body)的「運動」狀態該如何描述？



2.1 导读

- 一個剛體(Rigid body)的「運動」狀態該如何描述？
  - 利用各個DOF的微分，將位移和姿態（displacement / orientation）轉換到速度（velocity）和加速度（acceleration）等運動狀態

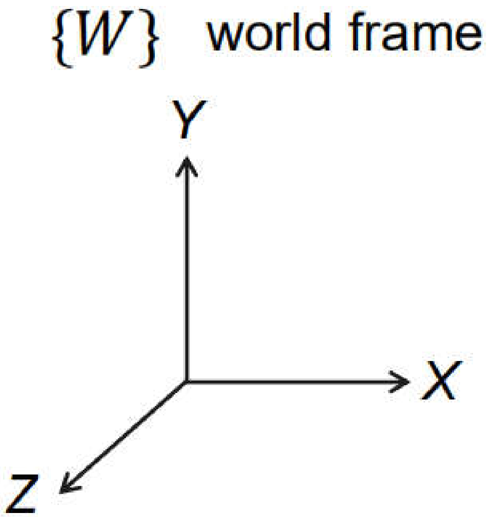


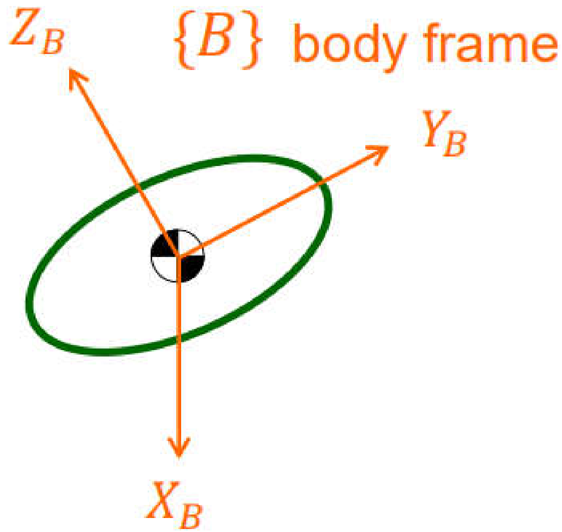
第二章 空间描述和变换

- 2.1 导读
- 2.2 移动
- 2.3 转动
- 2.4 旋转矩阵
- 2.5 旋转矩阵与转角
- 2.6 齐次变换矩阵
- 2.7 变换矩阵的运算法则

2.2 移动

- 移動：以向量（vector）$\vec{P}$ 來描述$\{B\}$的原點相對於$\{A\}$的狀態





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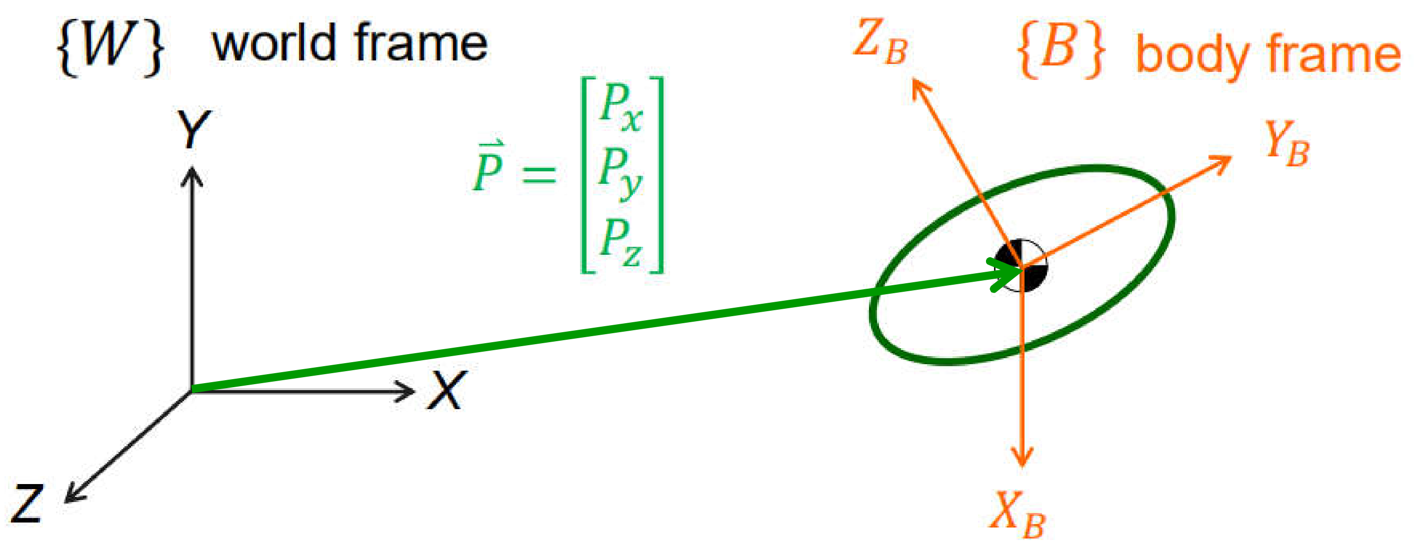

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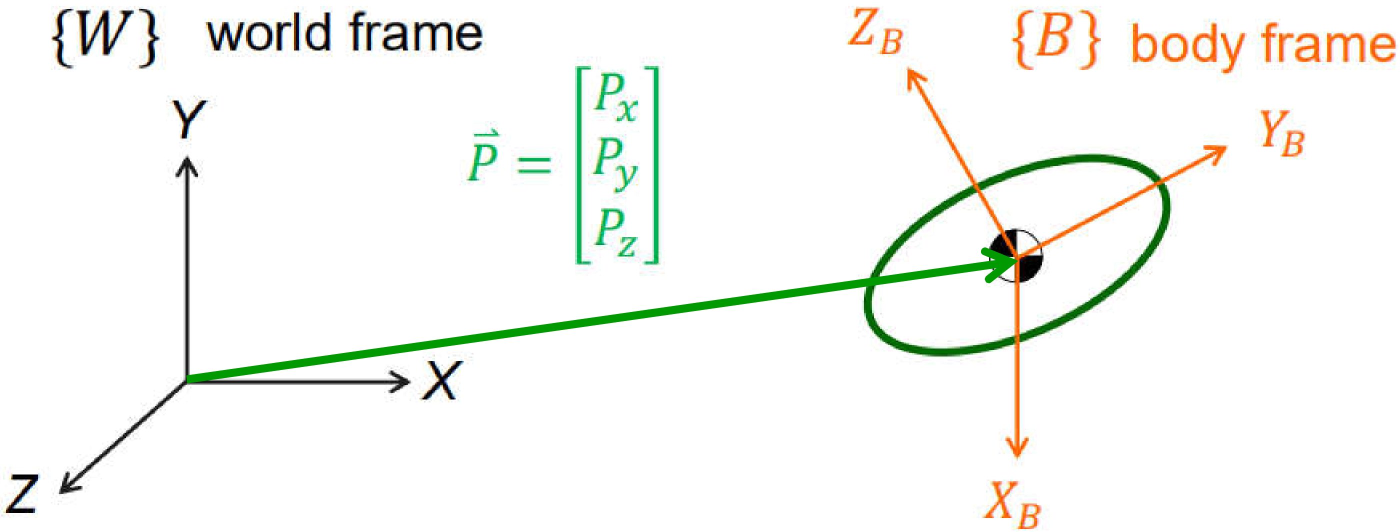

- Ex: $\vec{P} = \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} = \begin{bmatrix} 10 \\ 3 \\ 3 \end{bmatrix}$

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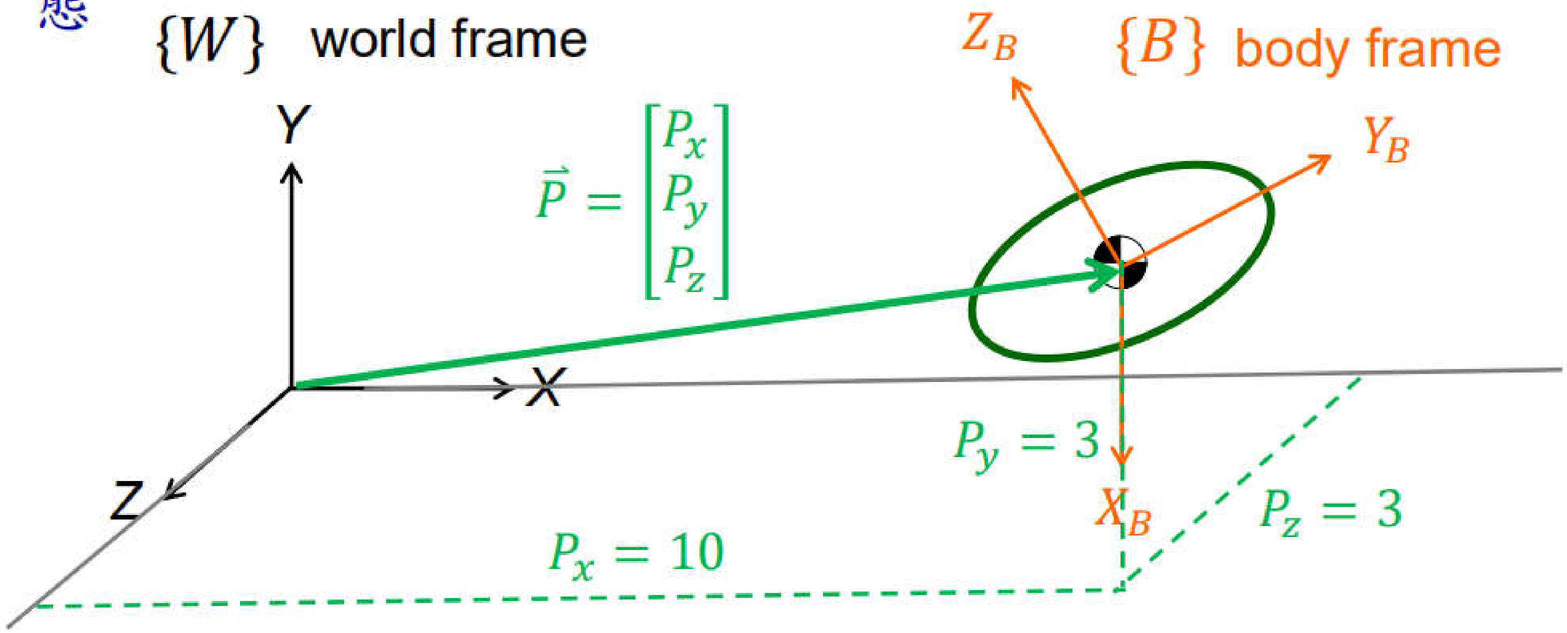


- Ex: $\vec{P} = \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} = \begin{bmatrix} 10 \\ 3 \\ 3 \end{bmatrix}$

2.2 移动

- 向量可表達空間關係的兩個方式

  - A position in space (i.e., position vector)

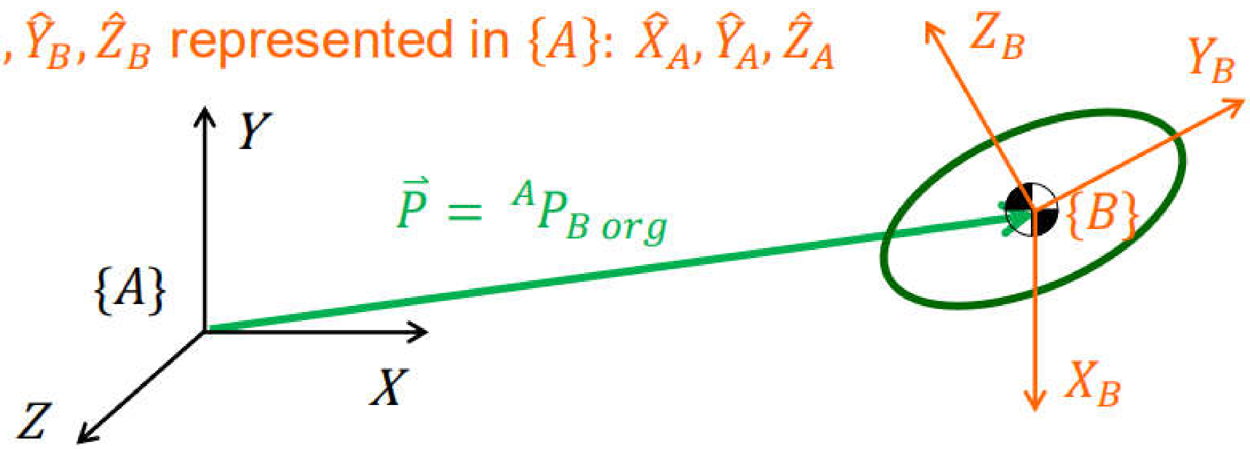
    如同前一頁內容，以此方式描述body frame原點

$$\vec{P} = \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} = {}^{A}P_{B\,org} = \text{origin of } \{B\} \text{ represented in } \{A\}$$

  - A vector (i.e., displacement, frame basis)

    以此方式表達body frame上principal axes的方向

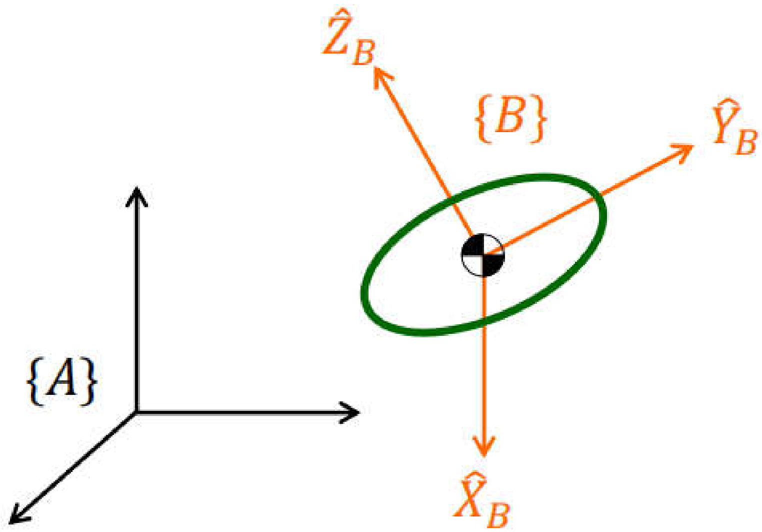
$\{B\}$: $\hat{X}_B, \hat{Y}_B, \hat{Z}_B$ represented in $\{A\}$: $\hat{X}_A, \hat{Y}_A, \hat{Z}_A$



第二章 空间描述和变换

2.3 转动

- 轉動：描述{B}相對於{A}之姿態---Rotation Matrix



2.3 转动

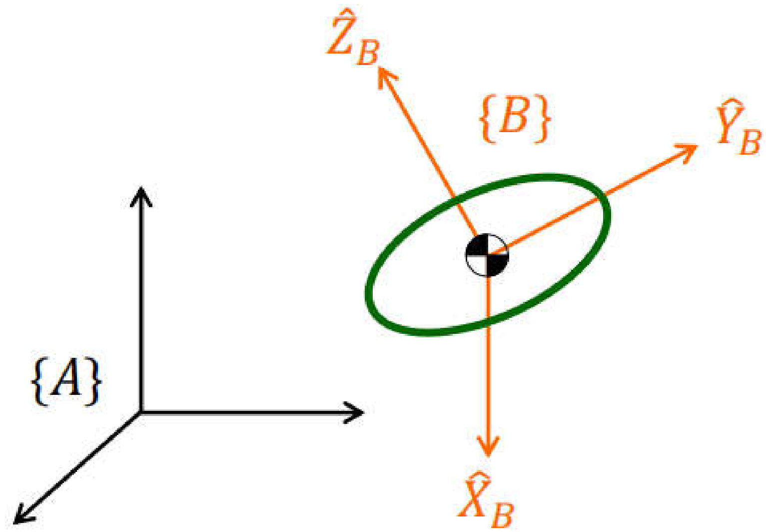
- 轉動：描述$\{B\}$相對於$\{A\}$之姿態---Rotation Matrix

$$
{}^{A}_{B}R = \begin{bmatrix} | & | & | \\ {}^{A}\hat{X}_B & {}^{A}\hat{Y}_B & {}^{A}\hat{Z}_B \\ | & | & | \end{bmatrix}
$$

B relative to A

"column vector"

R的三個columns即為frame $\{B\}$的$basis$: $\hat{X}_B, \hat{Y}_B, \hat{Z}_B$（由$\{A\}$看）



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- 轉動：描述$\{B\}$相對於$\{A\}$之姿態---Rotation Matrix

$$ {}^A_B R = \begin{bmatrix} | & | & | \\ {}^A\hat{X}_B & {}^A\hat{Y}_B & {}^A\hat{Z}_B \\ | & | & | \end{bmatrix} $$

B relative to A

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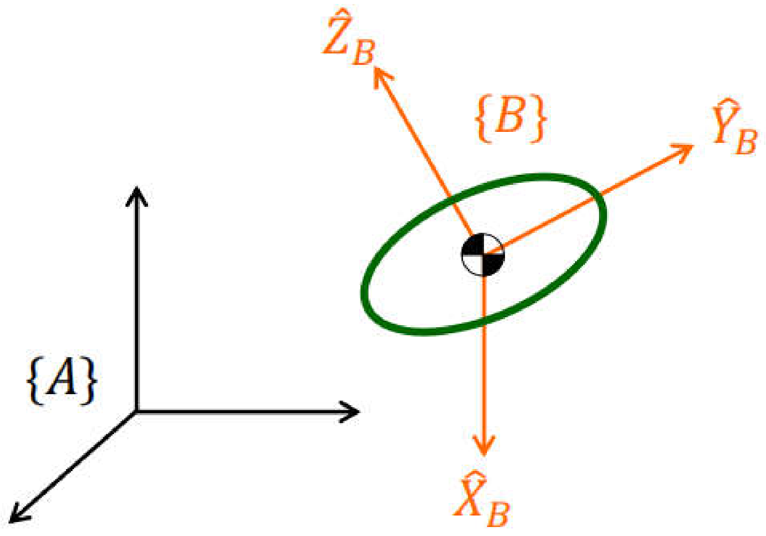
R的三個columns即為frame $\{B\}$的$basis$: $\hat{X}_B, \hat{Y}_B, \hat{Z}_B$（由$\{A\}$看）

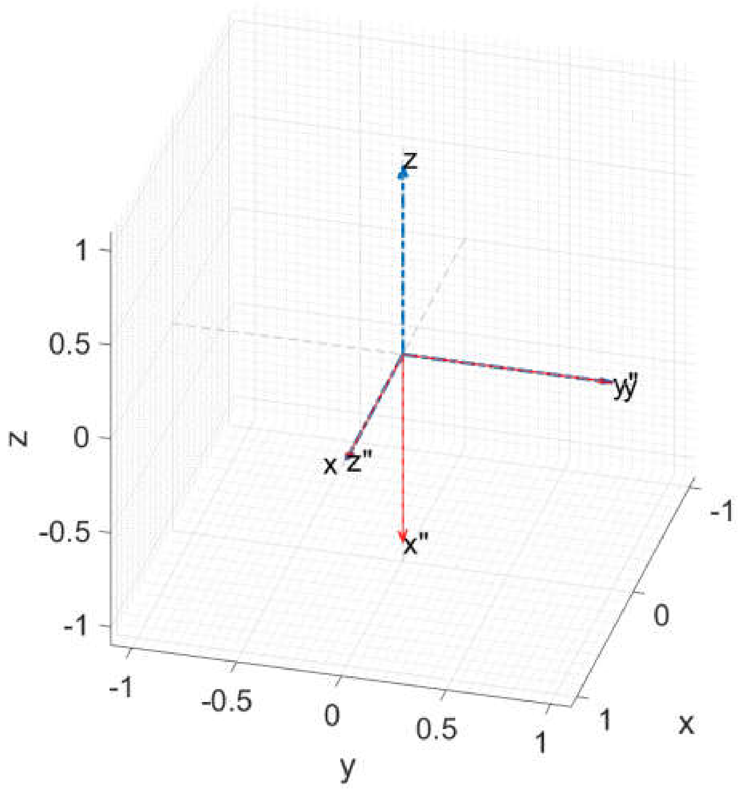
$$ = \begin{bmatrix} \hat{X}_B \cdot \hat{X}_A & \hat{Y}_B \cdot \hat{X}_A & \hat{Z}_B \cdot \hat{X}_A \\ \hat{X}_B \cdot \hat{Y}_A & \hat{Y}_B \cdot \hat{Y}_A & \hat{Z}_B \cdot \hat{Y}_A \\ \hat{X}_B \cdot \hat{Z}_A & \hat{Y}_B \cdot \hat{Z}_A & \hat{Z}_B \cdot \hat{Z}_A \end{bmatrix} $$

"direct cosines"



2.3 转动

- Ex: $\{B\}$相對於$\{A\}$之姿態 ${}^{A}_{B}R=?$



藍虛線: World Frame $\{A\}$

紅實線: Body Frame $\{B\}$

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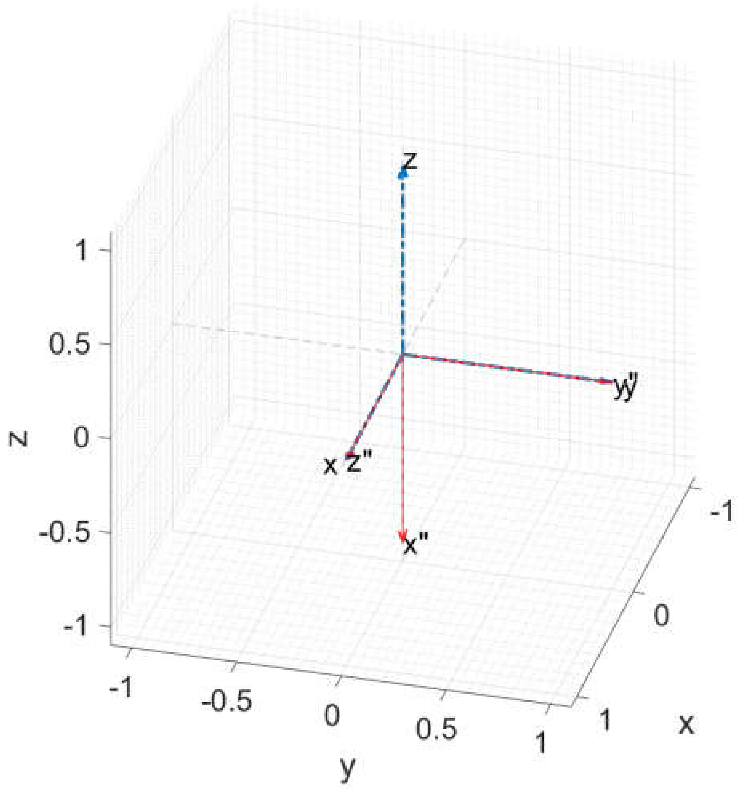
- Ex: $\{B\}$相對於$\{A\}$之姿態 ${}^{A}_{B}R = ?$



藍虛線: World Frame $\{A\}$

紅實線: Body Frame $\{B\}$

$\{B\}$的x″軸為$\{A\}$的z軸反向 $\Rightarrow {}^{A}\hat{X}_B = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$

$\{B\}$的y″軸與$\{A\}$的y軸重疊 $\Rightarrow {}^{A}\hat{Y}_B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\{B\}$的z″軸與$\{A\}$的x軸重疊 $\Rightarrow {}^{A}\hat{Z}_B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

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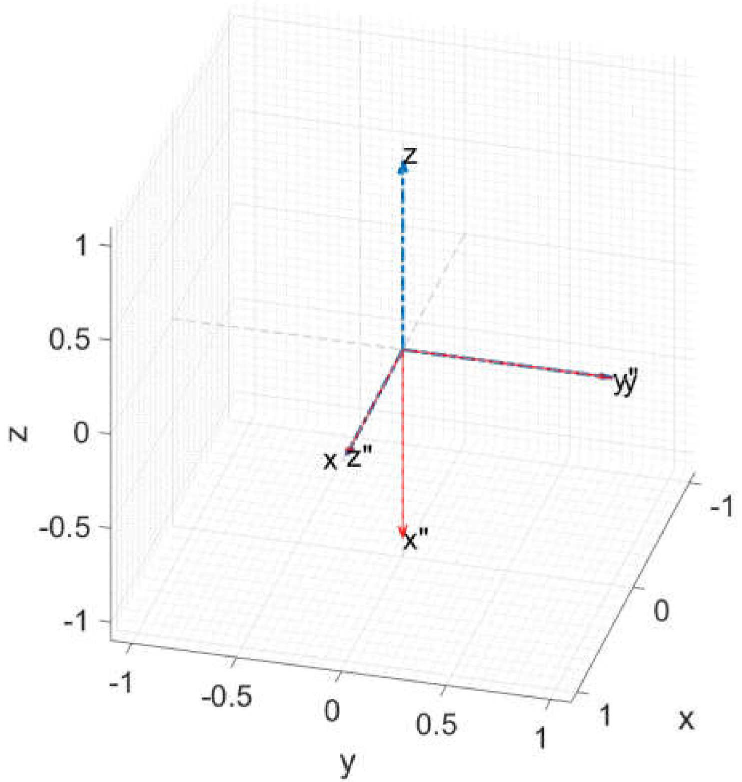
2.3 转动

- Ex: $\{B\}$相對於$\{A\}$之姿態 ${}^{A}_{B}R=?$



藍虛線: World Frame $\{A\}$

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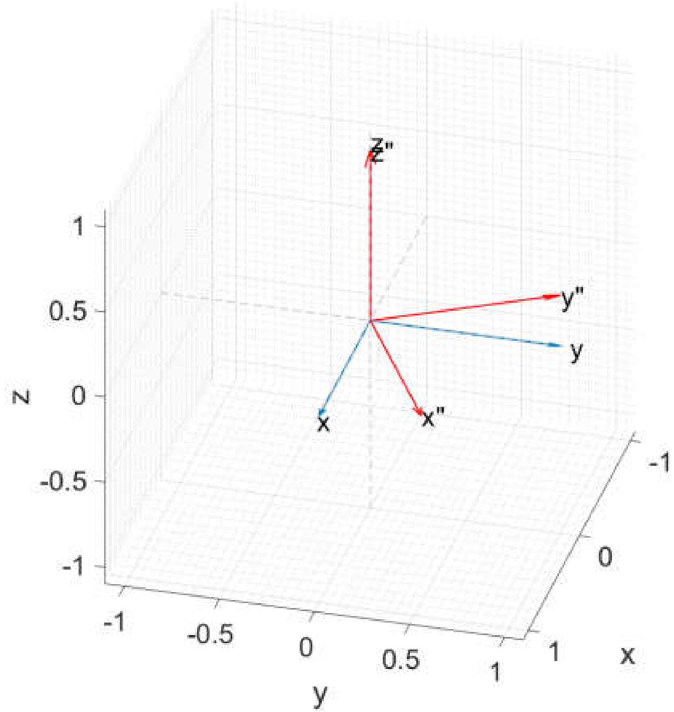
$\{B\}$的y″軸與$\{A\}$的y軸重疊 $\Rightarrow {}^{A}\hat{Y}_B=\begin{bmatrix}0\\1\\0\end{bmatrix}$

$\{B\}$的z″軸與$\{A\}$的x軸重疊 $\Rightarrow {}^{A}\hat{Z}_B=\begin{bmatrix}1\\0\\0\end{bmatrix}$

因此，$\{B\}$相對於$\{A\}$之姿態： ${}^{A}_{B}R=\begin{bmatrix}| & | & |\\ {}^{A}\hat{X}_B & {}^{A}\hat{Y}_B & {}^{A}\hat{Z}_B\\ | & | & |\end{bmatrix}=\begin{bmatrix}0 & 0 & 1\\0 & 1 & 0\\-1 & 0 & 0\end{bmatrix}$

2.3 转动

- Ex: $\{B\}$相對於$\{A\}$之姿態 ${}^{A}_{B}R=?$

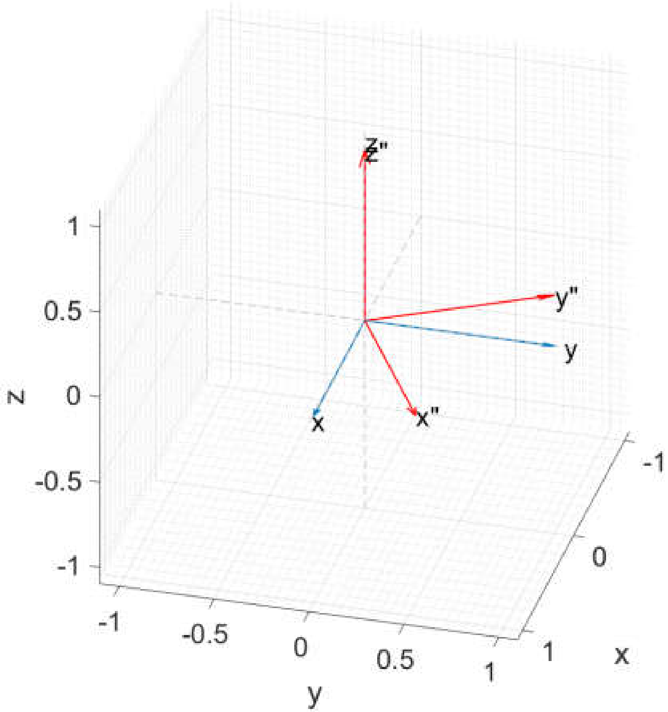


藍虛線: World Frame $\{A\}$

紅實線: Body Frame $\{B\}$

2.3 转动

- Ex: $\{B\}$相對於$\{A\}$之姿態 ${}^A_BR$ =?



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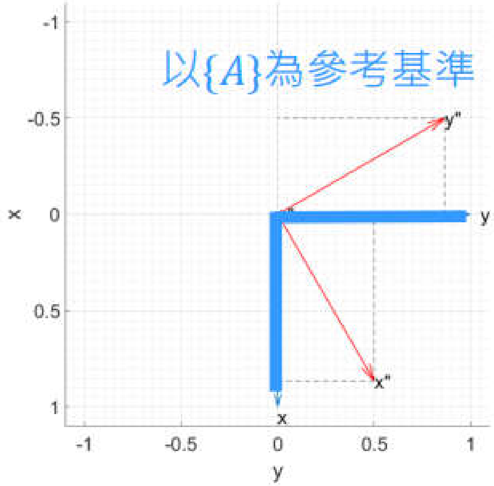

藍虛線: World Frame $\{A\}$

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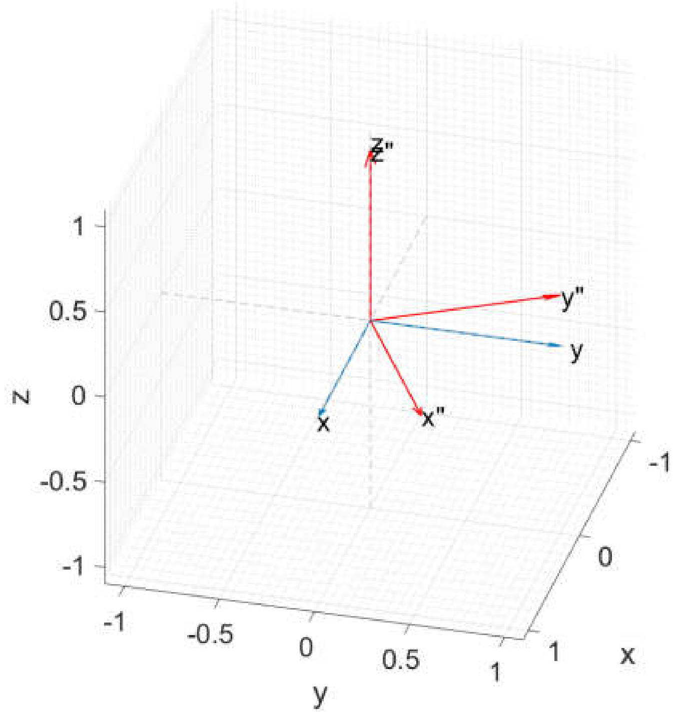
- Ex: $\{B\}$相對於$\{A\}$之姿態 ${}^{A}_{B}R = ?$



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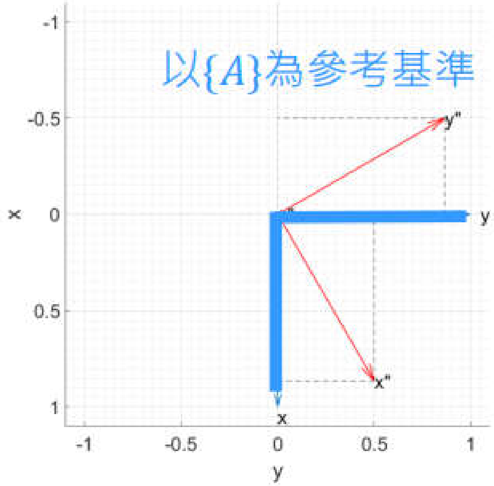


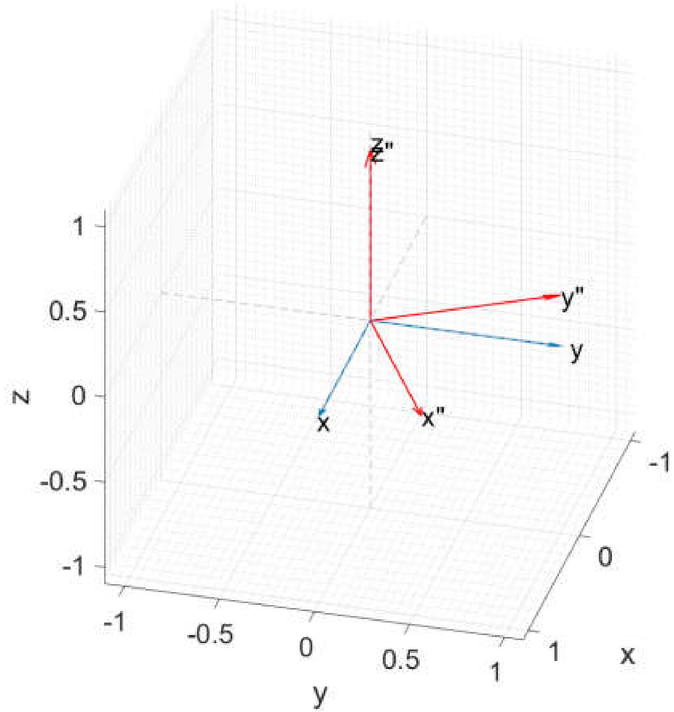
$$
{}^{A}\hat{X}_B = \begin{bmatrix} \hat{X}_B \cdot \hat{X}_A \\ \hat{X}_B \cdot \hat{Y}_A \\ \hat{X}_B \cdot \hat{Z}_A \end{bmatrix} = \begin{bmatrix} 0.866 \\ 0.5 \\ 0 \end{bmatrix}
$$

藍虛線: World Frame $\{A\}$

紅實線: Body Frame $\{B\}$

2.3 转动

- Ex: $\{B\}$相對於$\{A\}$之姿態 ${}^{A}_{B}R = ?$



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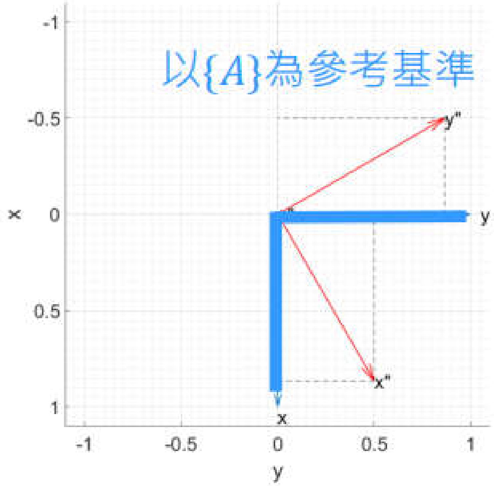

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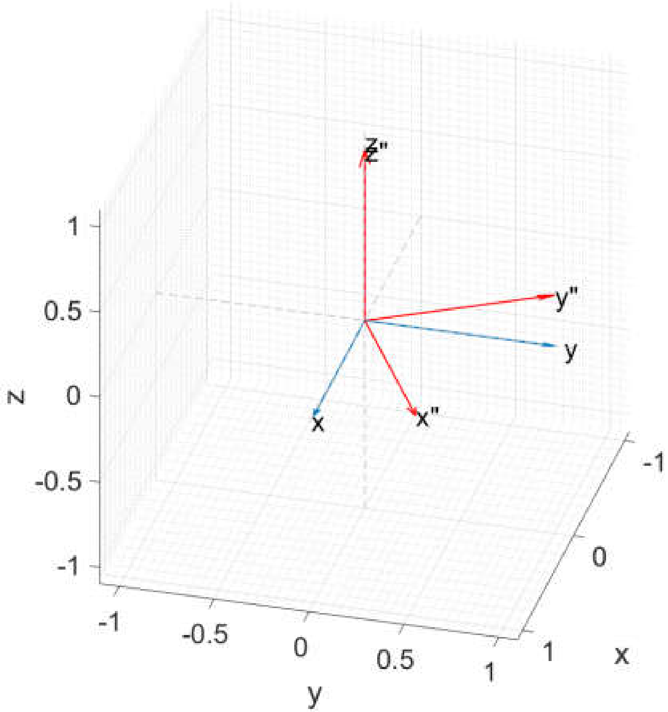
$$ {}^{A}\hat{Y}_B = \begin{bmatrix} \hat{Y}_B \cdot \hat{X}_A \\ \hat{Y}_B \cdot \hat{Y}_A \\ \hat{Y}_B \cdot \hat{Z}_A \end{bmatrix} = \begin{bmatrix} -0.5 \\ 0.866 \\ 0 \end{bmatrix} $$

藍虛線: World Frame $\{A\}$

紅實線: Body Frame $\{B\}$

2.3 转动

- Ex: $\{B\}$相對於$\{A\}$之姿態 ${}^{A}_{B}R = ?$



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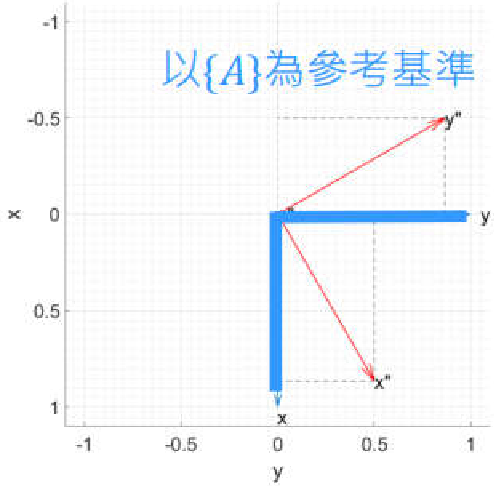

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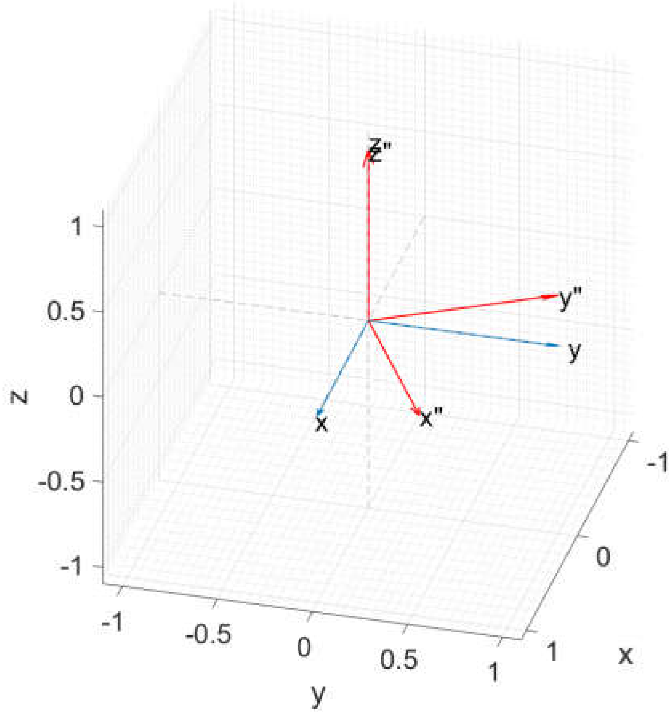
$$ {}^{A}\hat{Z}_B = \begin{bmatrix} \hat{Z}_B \cdot \hat{X}_A \\ \hat{Z}_B \cdot \hat{Y}_A \\ \hat{Z}_B \cdot \hat{Z}_A \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} $$

藍虛線: World Frame $\{A\}$

紅實線: Body Frame $\{B\}$

2.3 转动

- Ex: $\{B\}$相對於$\{A\}$之姿態 ${}^{A}_{B}R = ?$



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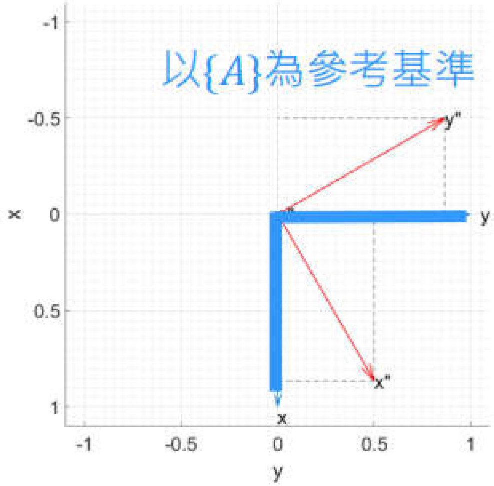


藍虛線: World Frame $\{A\}$

紅實線: Body Frame $\{B\}$

$$
{}^{A}\hat{X}_B = \begin{bmatrix} \hat{X}_B \cdot \hat{X}_A \\ \hat{X}_B \cdot \hat{Y}_A \\ \hat{X}_B \cdot \hat{Z}_A \end{bmatrix} = \begin{bmatrix} 0.866 \\ 0.5 \\ 0 \end{bmatrix}
$$

$$
{}^{A}\hat{Y}_B = \begin{bmatrix} \hat{Y}_B \cdot \hat{X}_A \\ \hat{Y}_B \cdot \hat{Y}_A \\ \hat{Y}_B \cdot \hat{Z}_A \end{bmatrix} = \begin{bmatrix} -0.5 \\ 0.866 \\ 0 \end{bmatrix}
$$

$$
{}^{A}\hat{Z}_B = \begin{bmatrix} \hat{Z}_B \cdot \hat{X}_A \\ \hat{Z}_B \cdot \hat{Y}_A \\ \hat{Z}_B \cdot \hat{Z}_A \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
$$

$\{B\}$相對於$\{A\}$之姿態：

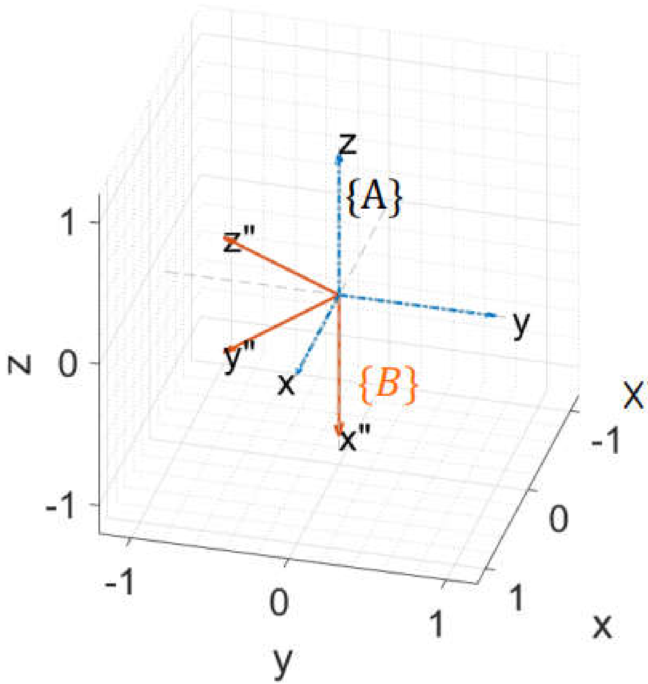
$$
{}^{A}_{B}R = \begin{bmatrix} 0.866 & -0.5 & 0 \\ 0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

2.3 转动

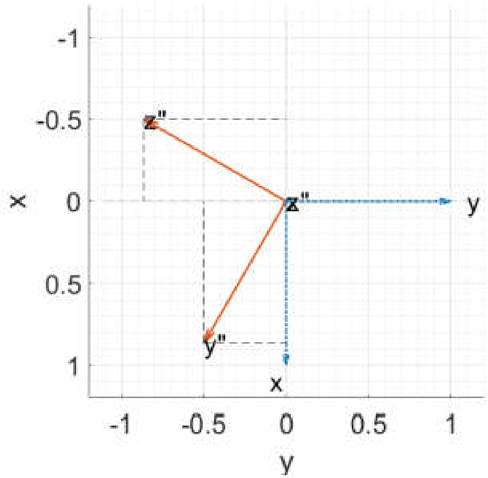
- In-video Quiz: $\{B\}$相對於$\{A\}$之姿態 ${}^{A}_{B}R = ?$

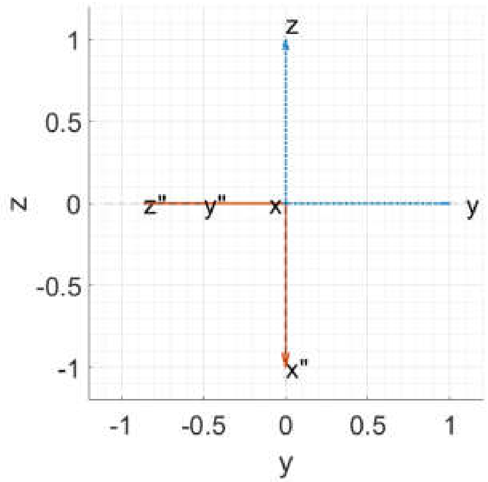
藍虛線: World Frame $\{A\}$

紅實線: Body Frame $\{B\}$



投影到
XY/YZ平面





A. $\begin{bmatrix} 0 & -0.866 & 0.5 \\ 0 & 0.5 & 0.866 \\ -1 & 0 & 0 \end{bmatrix}$

B. $\begin{bmatrix} 0 & 0.866 & -0.5 \\ 0 & -0.5 & -0.866 \\ -1 & 0 & 0 \end{bmatrix}$

C. $\begin{bmatrix} 0 & 0 & -1 \\ 0.866 & -0.5 & 0 \\ -0.5 & -0.866 & 0 \end{bmatrix}$

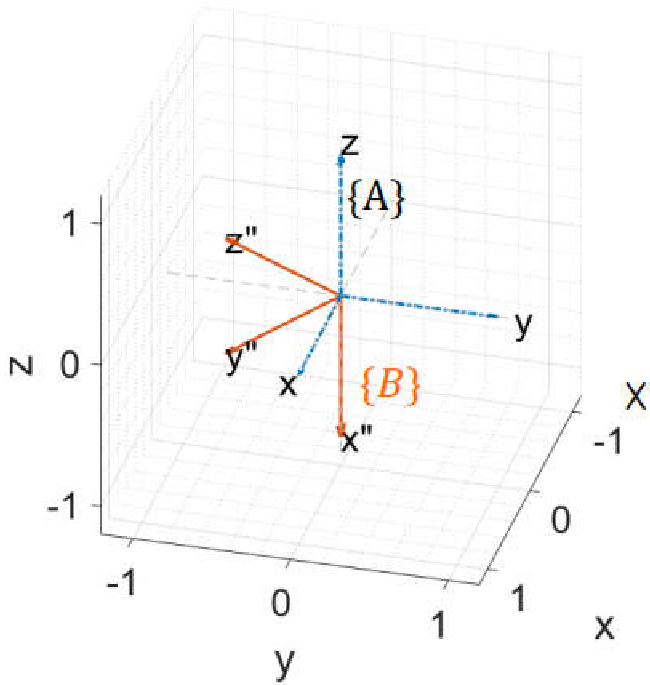
D. $\begin{bmatrix} 0 & 0 & -1 \\ -0.866 & 0.5 & 0 \\ 0.5 & 0.866 & 0 \end{bmatrix}$

2.3 转动

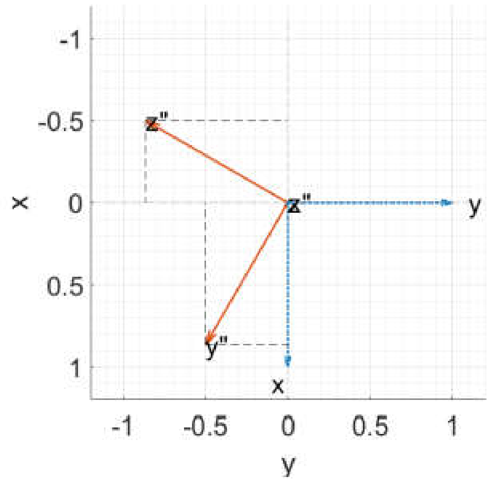
- In-video Quiz: $\{B\}$相對於$\{A\}$之姿態 ${}^{A}_{B}R = ?$

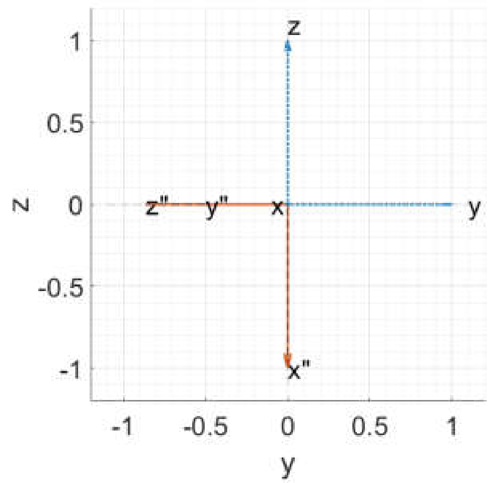
藍虛線: World Frame $\{A\}$

紅實線: Body Frame $\{B\}$



投影到
XY/YZ平面





A. $\begin{bmatrix} 0 & -0.866 & 0.5 \\ 0 & 0.5 & 0.866 \\ -1 & 0 & 0 \end{bmatrix}$

B. $\begin{bmatrix} 0 & 0.866 & -0.5 \\ 0 & -0.5 & -0.866 \\ -1 & 0 & 0 \end{bmatrix}$

C. $\begin{bmatrix} 0 & 0 & -1 \\ 0.866 & -0.5 & 0 \\ -0.5 & -0.866 & 0 \end{bmatrix}$

D. $\begin{bmatrix} 0 & 0 & -1 \\ -0.866 & 0.5 & 0 \\ 0.5 & 0.866 & 0 \end{bmatrix}$

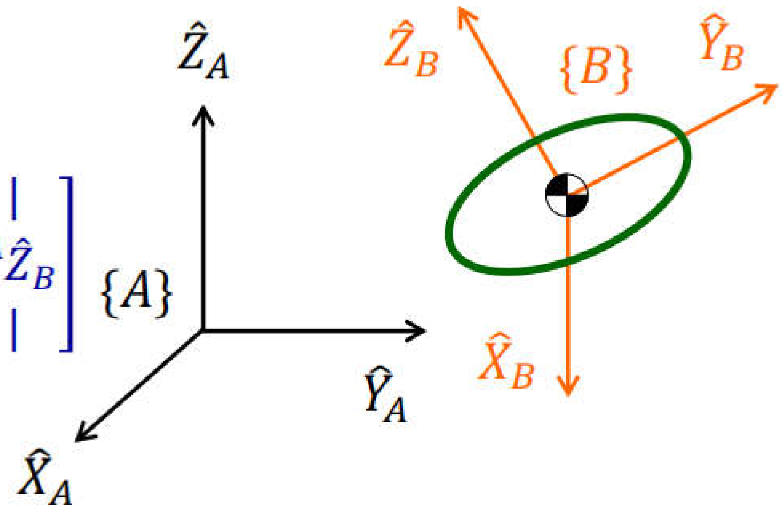
第二章 空间描述和变换

- 2.1 导读
- 2.2 移动
- 2.3 转动
- **2.4 旋转矩阵**
- 2.5 旋转矩阵与转角
- 2.6 齐次变换矩阵
- 2.7 变换矩阵的运算法则

2.4 旋转矩阵

- 特性

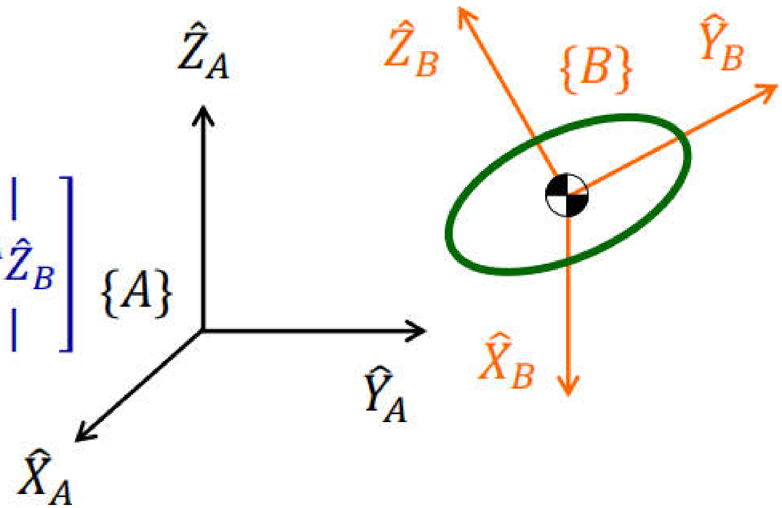
$$ {}^A_BR = \begin{bmatrix} \hat{X}_B\cdot\hat{X}_A & \hat{Y}_B\cdot\hat{X}_A & \hat{Z}_B\cdot\hat{X}_A \\ \hat{X}_B\cdot\hat{Y}_A & \hat{Y}_B\cdot\hat{Y}_A & \hat{Z}_B\cdot\hat{Y}_A \\ \hat{X}_B\cdot\hat{Z}_A & \hat{Y}_B\cdot\hat{Z}_A & \hat{Z}_B\cdot\hat{Z}_A \end{bmatrix} = \begin{bmatrix} | & | & | \\ {}^A\hat{X}_B & {}^A\hat{Y}_B & {}^A\hat{Z}_B \\ | & | & | \end{bmatrix} $$



2.4 旋转矩阵

- 特性

$$ {}^A_BR = \begin{bmatrix} \hat{X}_B \cdot \hat{X}_A & \hat{Y}_B \cdot \hat{X}_A & \hat{Z}_B \cdot \hat{X}_A \\ \hat{X}_B \cdot \hat{Y}_A & \hat{Y}_B \cdot \hat{Y}_A & \hat{Z}_B \cdot \hat{Y}_A \\ \hat{X}_B \cdot \hat{Z}_A & \hat{Y}_B \cdot \hat{Z}_A & \hat{Z}_B \cdot \hat{Z}_A \end{bmatrix} = \begin{bmatrix} | & | & | \\ {}^A\hat{X}_B & {}^A\hat{Y}_B & {}^A\hat{Z}_B \\ | & | & | \end{bmatrix} $$



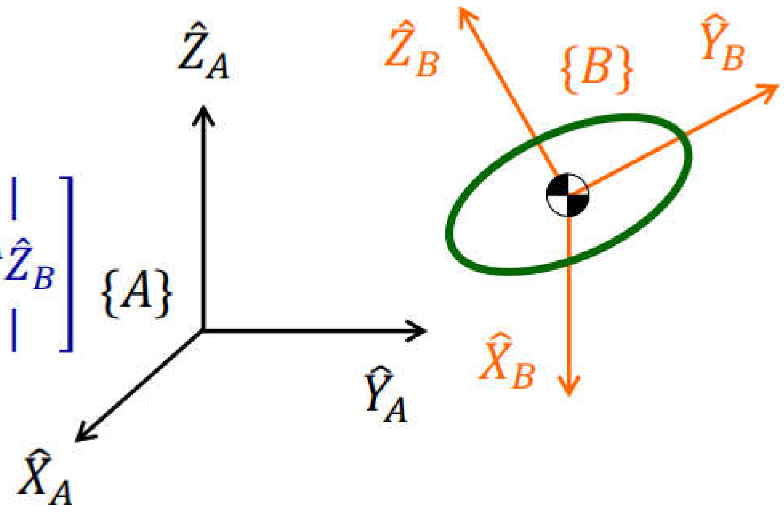
前後向量互換

$$ = \begin{bmatrix} \hat{X}_A \cdot \hat{X}_B & \hat{X}_A \cdot \hat{Y}_B & \hat{X}_A \cdot \hat{Z}_B \\ \hat{Y}_A \cdot \hat{X}_B & \hat{Y}_A \cdot \hat{Y}_B & \hat{Y}_A \cdot \hat{Z}_B \\ \hat{Z}_A \cdot \hat{X}_B & \hat{Z}_A \cdot \hat{Y}_B & \hat{Z}_A \cdot \hat{Z}_B \end{bmatrix} = \begin{bmatrix} - & {}^B\hat{X}_A{}^T & - \\ - & {}^B\hat{Y}_A{}^T & - \\ - & {}^B\hat{Z}_A{}^T & - \end{bmatrix} $$

2.4 旋转矩阵

- 特性

$$ {}^{A}_{B}R = \begin{bmatrix} \hat{X}_B \cdot \hat{X}_A & \hat{Y}_B \cdot \hat{X}_A & \hat{Z}_B \cdot \hat{X}_A \\ \hat{X}_B \cdot \hat{Y}_A & \hat{Y}_B \cdot \hat{Y}_A & \hat{Z}_B \cdot \hat{Y}_A \\ \hat{X}_B \cdot \hat{Z}_A & \hat{Y}_B \cdot \hat{Z}_A & \hat{Z}_B \cdot \hat{Z}_A \end{bmatrix} = \begin{bmatrix} | & | & | \\ {}^{A}\hat{X}_B & {}^{A}\hat{Y}_B & {}^{A}\hat{Z}_B \\ | & | & | \end{bmatrix} $$



前後向量互換

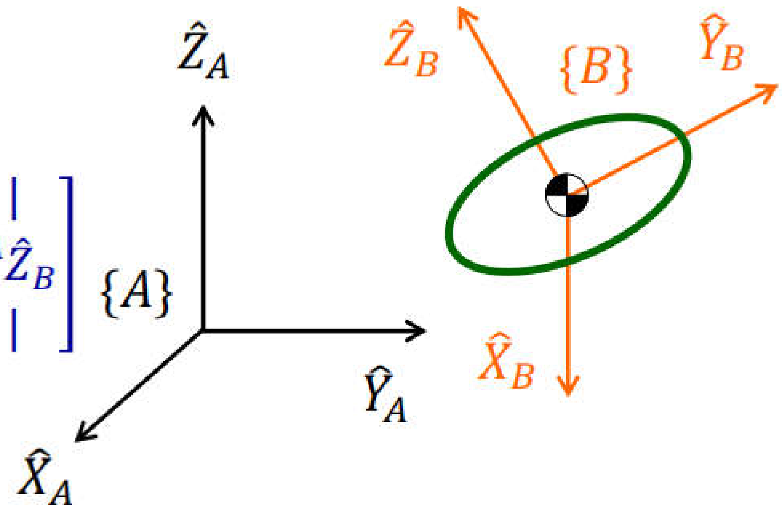
$$ = \begin{bmatrix} \hat{X}_A \cdot \hat{X}_B & \hat{X}_A \cdot \hat{Y}_B & \hat{X}_A \cdot \hat{Z}_B \\ \hat{Y}_A \cdot \hat{X}_B & \hat{Y}_A \cdot \hat{Y}_B & \hat{Y}_A \cdot \hat{Z}_B \\ \hat{Z}_A \cdot \hat{X}_B & \hat{Z}_A \cdot \hat{Y}_B & \hat{Z}_A \cdot \hat{Z}_B \end{bmatrix} = \begin{bmatrix} - & {}^{B}\hat{X}_A{}^{T} & - \\ - & {}^{B}\hat{Y}_A{}^{T} & - \\ - & {}^{B}\hat{Z}_A{}^{T} & - \end{bmatrix} $$

$$ = \begin{bmatrix} | & | & | \\ {}^{B}\hat{X}_A & {}^{B}\hat{Y}_A & {}^{B}\hat{Z}_A \\ | & | & | \end{bmatrix}^{T} = {}^{B}_{A}R^{T} $$

2.4 旋转矩阵

❑ 特性

$$ {}^A_BR = \begin{bmatrix} \hat{X}_B\cdot\hat{X}_A & \hat{Y}_B\cdot\hat{X}_A & \hat{Z}_B\cdot\hat{X}_A \\ \hat{X}_B\cdot\hat{Y}_A & \hat{Y}_B\cdot\hat{Y}_A & \hat{Z}_B\cdot\hat{Y}_A \\ \hat{X}_B\cdot\hat{Z}_A & \hat{Y}_B\cdot\hat{Z}_A & \hat{Z}_B\cdot\hat{Z}_A \end{bmatrix} = \begin{bmatrix} | & | & | \\ {}^A\hat{X}_B & {}^A\hat{Y}_B & {}^A\hat{Z}_B \\ | & | & | \end{bmatrix} $$



前後向量互換

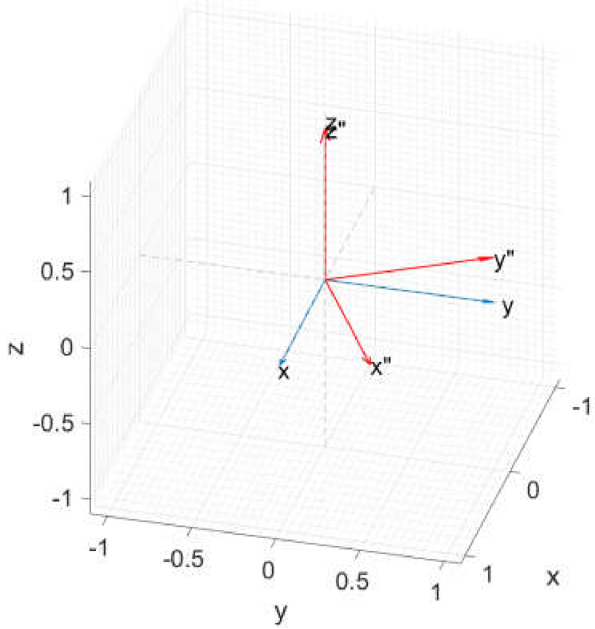
$$ = \begin{bmatrix} \hat{X}_A\cdot\hat{X}_B & \hat{X}_A\cdot\hat{Y}_B & \hat{X}_A\cdot\hat{Z}_B \\ \hat{Y}_A\cdot\hat{X}_B & \hat{Y}_A\cdot\hat{Y}_B & \hat{Y}_A\cdot\hat{Z}_B \\ \hat{Z}_A\cdot\hat{X}_B & \hat{Z}_A\cdot\hat{Y}_B & \hat{Z}_A\cdot\hat{Z}_B \end{bmatrix} = \begin{bmatrix} - & {}^B\hat{X}_A{}^T & - \\ - & {}^B\hat{Y}_A{}^T & - \\ - & {}^B\hat{Z}_A{}^T & - \end{bmatrix} $$

$$ = \begin{bmatrix} | & | & | \\ {}^B\hat{X}_A & {}^B\hat{Y}_A & {}^B\hat{Z}_A \\ | & | & | \end{bmatrix}^T = {}^B_AR^T $$

⇨ ${}^A_BR = {}^B_AR^T$

2.4 旋转矩阵

- Ex: $\{A\}$相對於$\{B\}$之姿態 ${}^{B}_{A}R = ?$

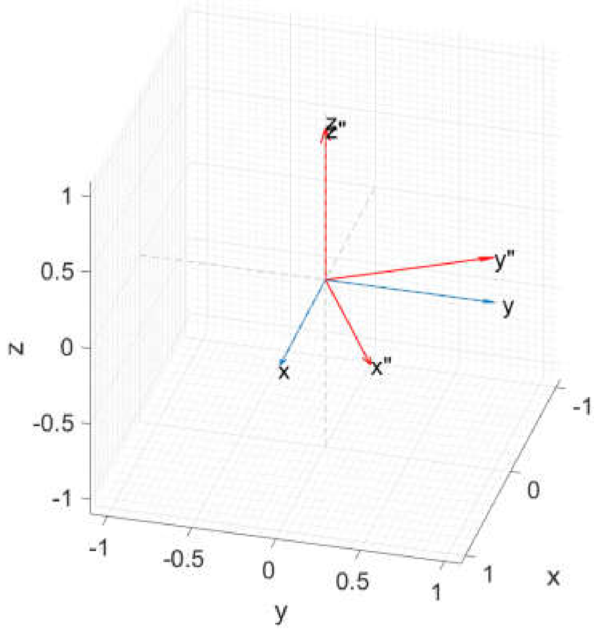


藍虛線: World Frame $\{A\}$

紅實線: Body Frame $\{B\}$

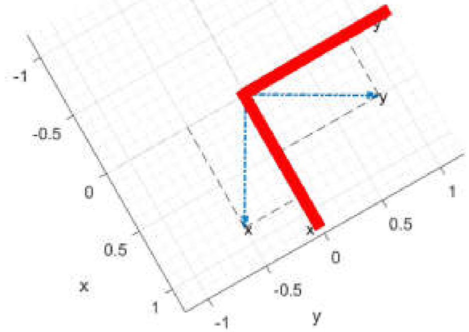
2.4 旋转矩阵

- Ex: $\{A\}$相對於$\{B\}$之姿態 ${}^{B}_{A}R$ =?



上視圖

以$\{B\}$為參考基準



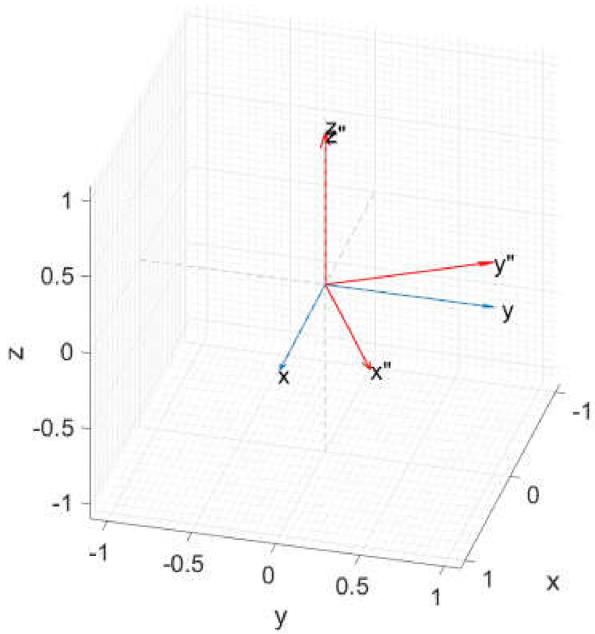
藍虛線: World Frame $\{A\}$

紅實線: Body Frame $\{B\}$

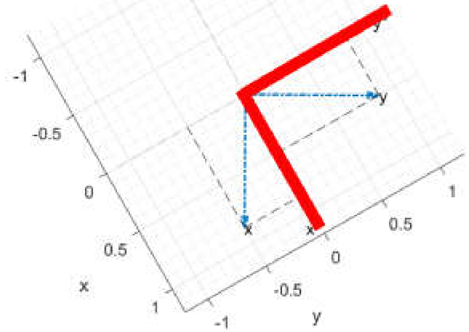
2.4 旋转矩阵

- Ex: $\{A\}$相對於$\{B\}$之姿態 ${}^{B}_{A}R=?$

以$\{B\}$為參考基準



上視圖



$$^{B}\hat{X}_A=\begin{bmatrix}\hat{X}_A\cdot\hat{X}_B\\\hat{Y}_A\cdot\hat{X}_B\\\hat{Z}_A\cdot\hat{X}_B\end{bmatrix}=\begin{bmatrix}0.866\\-0.5\\0\end{bmatrix}$$

$$^{B}\hat{Y}_A=\begin{bmatrix}\hat{X}_A\cdot\hat{Y}_B\\\hat{Y}_A\cdot\hat{Y}_B\\\hat{Z}_A\cdot\hat{Y}_B\end{bmatrix}=\begin{bmatrix}0.5\\0.866\\0\end{bmatrix}$$

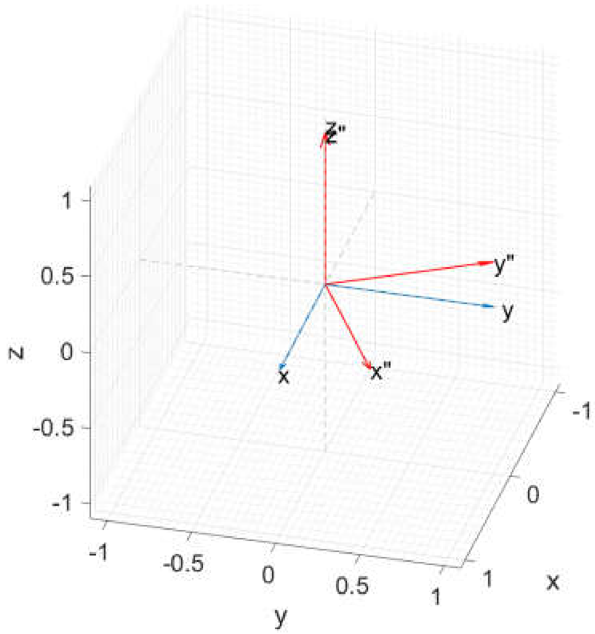
$$^{B}\hat{Z}_A=\begin{bmatrix}\hat{X}_A\cdot\hat{Z}_B\\\hat{Y}_A\cdot\hat{Z}_B\\\hat{Z}_A\cdot\hat{Z}_B\end{bmatrix}=\begin{bmatrix}0\\0\\1\end{bmatrix}$$

藍虛線: World Frame $\{A\}$

紅實線: Body Frame $\{B\}$

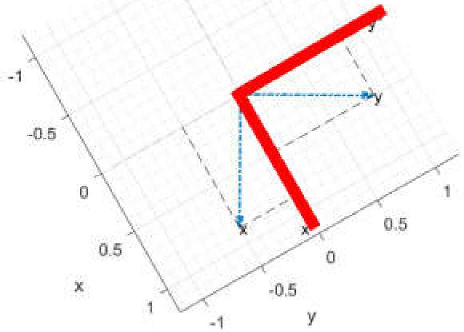
2.4 旋转矩阵

- Ex: $\{A\}$相對於$\{B\}$之姿態 ${}^{B}_{A}R = ?$



上視圖

以$\{B\}$為參考基準



藍虛線: World Frame $\{A\}$

紅實線: Body Frame $\{B\}$

$$^{B}\hat{X}_A = \begin{bmatrix}\hat{X}_A\cdot\hat{X}_B\\ \hat{Y}_A\cdot\hat{X}_B\\ \hat{Z}_A\cdot\hat{X}_B\end{bmatrix} = \begin{bmatrix}0.866\\ -0.5\\ 0\end{bmatrix}$$

$$^{B}\hat{Y}_A = \begin{bmatrix}\hat{X}_A\cdot\hat{Y}_B\\ \hat{Y}_A\cdot\hat{Y}_B\\ \hat{Z}_A\cdot\hat{Y}_B\end{bmatrix} = \begin{bmatrix}0.5\\ 0.866\\ 0\end{bmatrix}$$

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$\{A\}$相對於$\{B\}$之姿態：

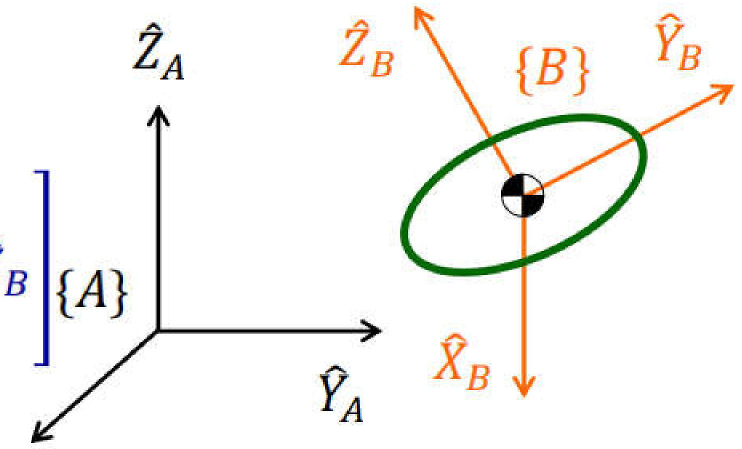
$$^{B}_{A}R = \begin{bmatrix}0.866 & 0.5 & 0\\ -0.5 & 0.866 & 0\\ 0 & 0 & 1\end{bmatrix} = {}^{A}_{B}R^{T}$$

${}^{A}_{B}R$，「轉動 -3」頁面結果

2.4 旋转矩阵

- 特性

$$ {}_B^A R^T \, {}_B^A R = \begin{bmatrix} | & | & | \\ {}^A\hat{X}_B & {}^A\hat{Y}_B & {}^A\hat{Z}_B \\ | & | & | \end{bmatrix}^T \begin{bmatrix} | & | & | \\ {}^A\hat{X}_B & {}^A\hat{Y}_B & {}^A\hat{Z}_B \\ | & | & | \end{bmatrix} $$

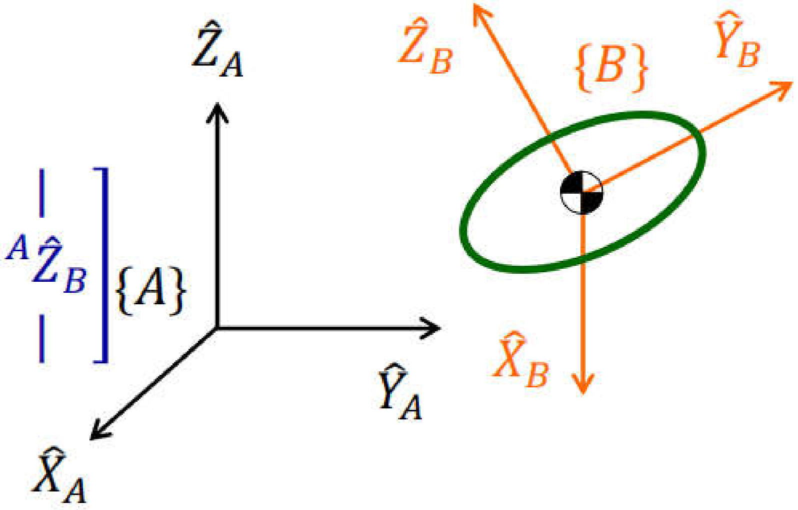


2.4 旋转矩阵

特性

$$
{}^A_BR^T\,{}^A_BR = \begin{bmatrix} | & | & | \\ {}^A\hat{X}_B & {}^A\hat{Y}_B & {}^A\hat{Z}_B \\ | & | & | \end{bmatrix}^T \begin{bmatrix} | & | & | \\ {}^A\hat{X}_B & {}^A\hat{Y}_B & {}^A\hat{Z}_B \\ | & | & | \end{bmatrix}
$$

$$
= \begin{bmatrix} - & {}^A\hat{X}_B{}^T & - \\ - & {}^A\hat{Y}_B{}^T & - \\ - & {}^A\hat{Z}_B{}^T & - \end{bmatrix} \begin{bmatrix} | & | & | \\ {}^A\hat{X}_B & {}^A\hat{Y}_B & {}^A\hat{Z}_B \\ | & | & | \end{bmatrix}
$$



2.4 旋转矩阵

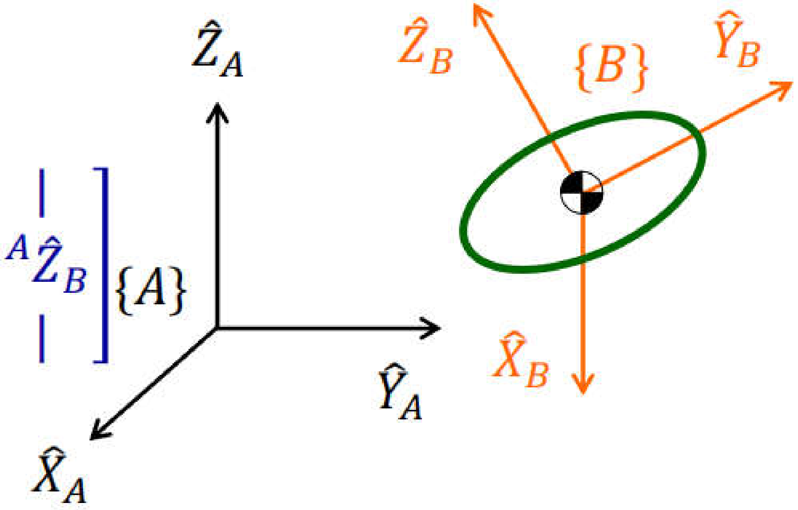
- 特性

$$ {}^A_B R^T \, {}^A_B R = \begin{bmatrix} | & | & | \\ {}^A\hat{X}_B & {}^A\hat{Y}_B & {}^A\hat{Z}_B \\ | & | & | \end{bmatrix}^T \begin{bmatrix} | & | & | \\ {}^A\hat{X}_B & {}^A\hat{Y}_B & {}^A\hat{Z}_B \\ | & | & | \end{bmatrix} $$

$$ = \begin{bmatrix} - & {}^A\hat{X}_B{}^T & - \\ - & {}^A\hat{Y}_B{}^T & - \\ - & {}^A\hat{Z}_B{}^T & - \end{bmatrix} \begin{bmatrix} | & | & | \\ {}^A\hat{X}_B & {}^A\hat{Y}_B & {}^A\hat{Z}_B \\ | & | & | \end{bmatrix} $$

$$ = I_3 $$

3x3 identity matrix



2.4 旋转矩阵

特性

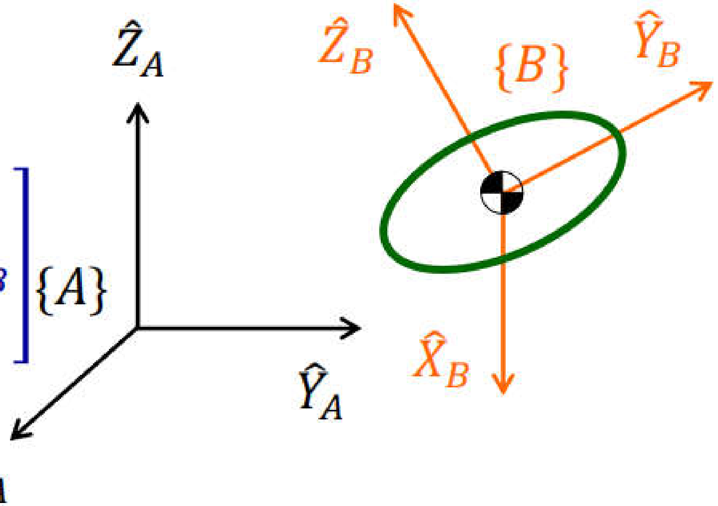
$$ {}^A_BR^T\,{}^A_BR = \begin{bmatrix} | & | & | \\ {}^A\hat{X}_B & {}^A\hat{Y}_B & {}^A\hat{Z}_B \\ | & | & | \end{bmatrix}^T \begin{bmatrix} | & | & | \\ {}^A\hat{X}_B & {}^A\hat{Y}_B & {}^A\hat{Z}_B \\ | & | & | \end{bmatrix} $$

$$ = \begin{bmatrix} - & {}^A\hat{X}_B{}^T & - \\ - & {}^A\hat{Y}_B{}^T & - \\ - & {}^A\hat{Z}_B{}^T & - \end{bmatrix} \begin{bmatrix} | & | & | \\ {}^A\hat{X}_B & {}^A\hat{Y}_B & {}^A\hat{Z}_B \\ | & | & | \end{bmatrix} $$

$$ = I_3 $$

3x3 identity matrix

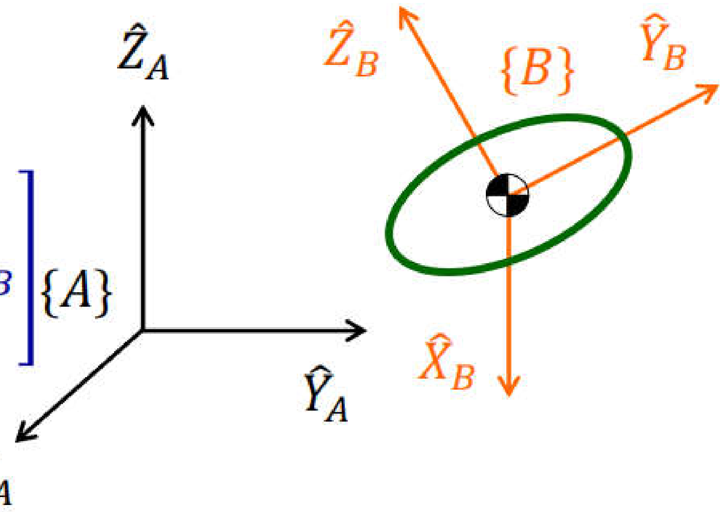
$$ = {}^A_BR^{-1}\,{}^A_BR $$



2.4 旋转矩阵

- 特性

$$ {}^{A}_{B}R^{T}\,{}^{A}_{B}R = \begin{bmatrix} | & | & | \\ {}^{A}\hat{X}_B & {}^{A}\hat{Y}_B & {}^{A}\hat{Z}_B \\ | & | & | \end{bmatrix}^{T} \begin{bmatrix} | & | & | \\ {}^{A}\hat{X}_B & {}^{A}\hat{Y}_B & {}^{A}\hat{Z}_B \\ | & | & | \end{bmatrix} $$



$$ = \begin{bmatrix} - & {}^{A}\hat{X}_B{}^{T} & - \\ - & {}^{A}\hat{Y}_B{}^{T} & - \\ - & {}^{A}\hat{Z}_B{}^{T} & - \end{bmatrix} \begin{bmatrix} | & | & | \\ {}^{A}\hat{X}_B & {}^{A}\hat{Y}_B & {}^{A}\hat{Z}_B \\ | & | & | \end{bmatrix} $$

$$ = I_3 $$

3x3 identity matrix

$$ = {}^{A}_{B}R^{-1}\,{}^{A}_{B}R $$

⇨ $ {}^{A}_{B}R^{T} = {}^{A}_{B}R^{-1} = {}^{B}_{A}R $

2.4 旋转矩阵

- A 3x3 orthogonal matrix Q $QQ^T = Q^T Q = I$
  - Always invertible $Q^{-1} = Q^T$

2.4 旋转矩阵

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    - Length = 1
    - Mutually perpendicular

2.4 旋转矩阵

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  - Rotation matrix (R)有9個數字，但上列兩個條件置入了6個constraints，所以R只有3個DOFs，與空間中轉動具有3 DOFs相符

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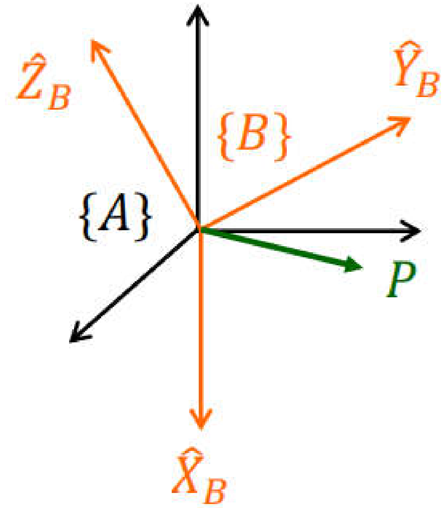
  - Rotation matrix (R)有9個數字，但上列兩個條件置入了6個 constraints，所以R只有3個DOFs，與空間中轉動具有3 DOFs相符
  - Determinant =1 (rotation); =-1 (reflection)

2.4 旋转矩阵

- Rotation matrix 除描述{B}相對於{A}之姿態，也可用於轉換向量之座標



original coordinate $^{B}P = {}^{B}P_x\hat{X}_B + {}^{B}P_y\hat{Y}_B + {}^{B}P_z\hat{Z}_B$

new coordinate $^{A}P = {}^{A}P_x\hat{X}_A + {}^{A}P_y\hat{Y}_A + {}^{A}P_z\hat{Z}_A$

2.4 旋转矩阵

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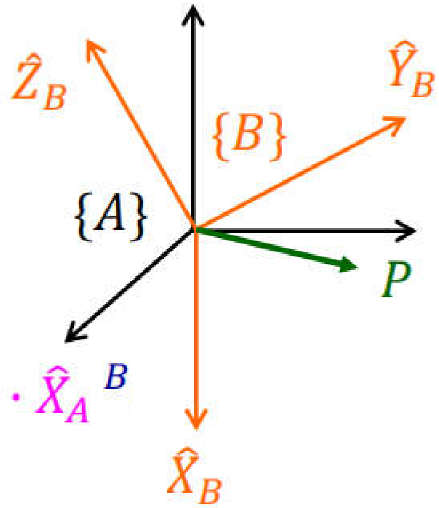

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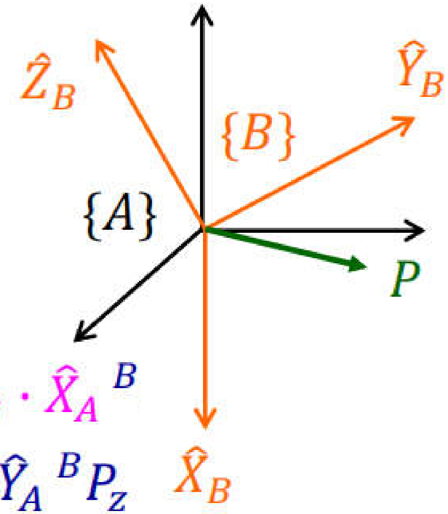
where $^{A}P_x = {}^{B}P \cdot \hat{X}_A = \hat{X}_B \cdot \hat{X}_A\, {}^{B}P_x + \hat{Y}_B \cdot \hat{X}_A\, {}^{B}P_y + \hat{Z}_B \cdot \hat{X}_A\, {}^{B}$

2.4 旋转矩阵

- Rotation matrix 除描述$\{B\}$相對於$\{A\}$之姿態，也可用於轉換向量之座標



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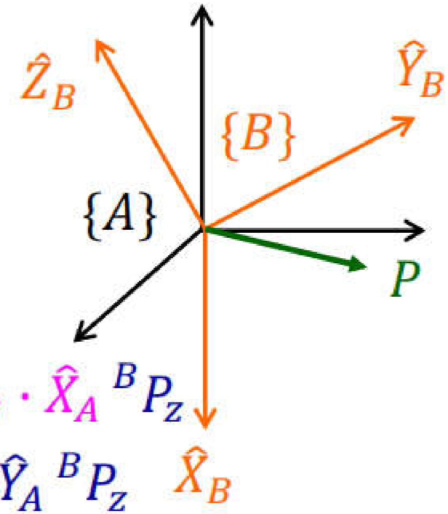
where $^{A}P_x = {}^{B}P\cdot\hat{X}_A = \hat{X}_B\cdot\hat{X}_A\,{}^{B}P_x + \hat{Y}_B\cdot\hat{X}_A\,{}^{B}P_y + \hat{Z}_B\cdot\hat{X}_A\,{}^{B}$

$$^{A}P_y = {}^{B}P\cdot\hat{Y}_A = \hat{X}_B\cdot\hat{Y}_A\,{}^{B}P_x + \hat{Y}_B\cdot\hat{Y}_A\,{}^{B}P_y + \hat{Z}_B\cdot\hat{Y}_A\,{}^{B}P_z$$

$$^{A}P_z = {}^{B}P\cdot\hat{Z}_A = \hat{X}_B\cdot\hat{Z}_A\,{}^{B}P_x + \hat{Y}_B\cdot\hat{Z}_A\,{}^{B}P_y + \hat{Z}_B\cdot\hat{Z}_A\,{}^{B}P_z$$

2.4 旋转矩阵

- Rotation matrix 除描述{B}相對於{A}之姿態，也可用於轉換向量之座標



original coordinate $^{B}P = {^{B}P_x}\hat{X}_B + {^{B}P_y}\hat{Y}_B + {^{B}P_z}\hat{Z}_B$

new coordinate $^{A}P = {^{A}P_x}\hat{X}_A + {^{A}P_y}\hat{Y}_A + {^{A}P_z}\hat{Z}_A$

where $^{A}P_x = {^{B}P} \cdot \hat{X}_A = \hat{X}_B \cdot \hat{X}_A \, {^{B}P_x} + \hat{Y}_B \cdot \hat{X}_A \, {^{B}P_y} + \hat{Z}_B \cdot \hat{X}_A \, {^{B}P_z}$

$$^{A}P_y = {^{B}P} \cdot \hat{Y}_A = \hat{X}_B \cdot \hat{Y}_A \, {^{B}P_x} + \hat{Y}_B \cdot \hat{Y}_A \, {^{B}P_y} + \hat{Z}_B \cdot \hat{Y}_A \, {^{B}P_z}$$

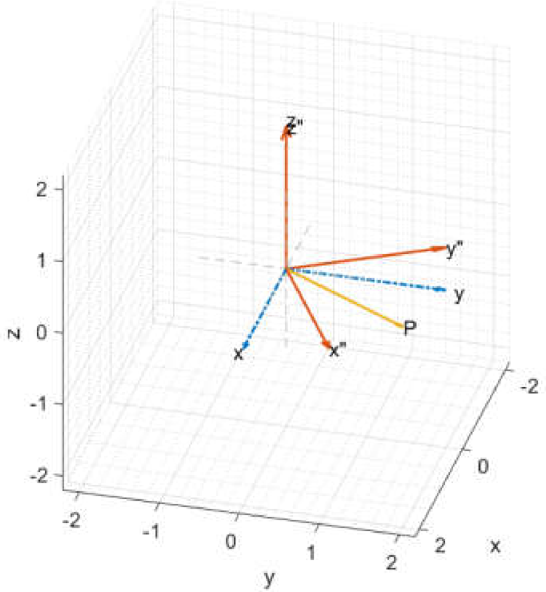
$$^{A}P_z = {^{B}P} \cdot \hat{Z}_A = \hat{X}_B \cdot \hat{Z}_A \, {^{B}P_x} + \hat{Y}_B \cdot \hat{Z}_A \, {^{B}P_y} + \hat{Z}_B \cdot \hat{Z}_A \, {^{B}P_z}$$

$$\Rightarrow \quad {^{A}P} = {^{A}\begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix}} = \begin{bmatrix} \hat{X}_B \cdot \hat{X}_A & \hat{Y}_B \cdot \hat{X}_A & \hat{Z}_B \cdot \hat{X}_A \\ \hat{X}_B \cdot \hat{Y}_A & \hat{Y}_B \cdot \hat{Y}_A & \hat{Z}_B \cdot \hat{Y}_A \\ \hat{X}_B \cdot \hat{Z}_A & \hat{Y}_B \cdot \hat{Z}_A & \hat{Z}_B \cdot \hat{Z}_A \end{bmatrix} {^{B}\begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix}} = {^{A}_{B}R} \, {^{B}P}$$

和「轉動-1」頁matrix相同，為rotation matrix

2.4 旋转矩阵

- Ex: 若$\{B\}$和$\{A\}$ 的相對狀態同「轉動 -3」頁面所示，假設 $^{B}P = [1.732 \quad 1 \quad 0]^T$，$^{A}P =?$



藍虛線: World Frame $\{A\}$

紅實線: Body Frame $\{B\}$

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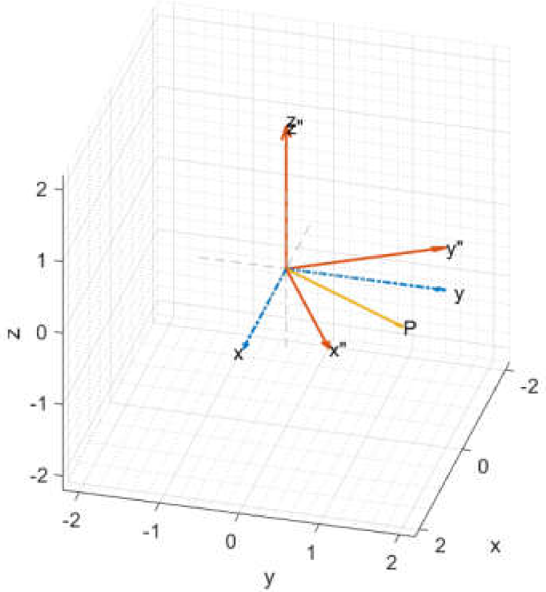

$${}^{A}P = {}^{A}_{B}R \, {}^{B}P$$

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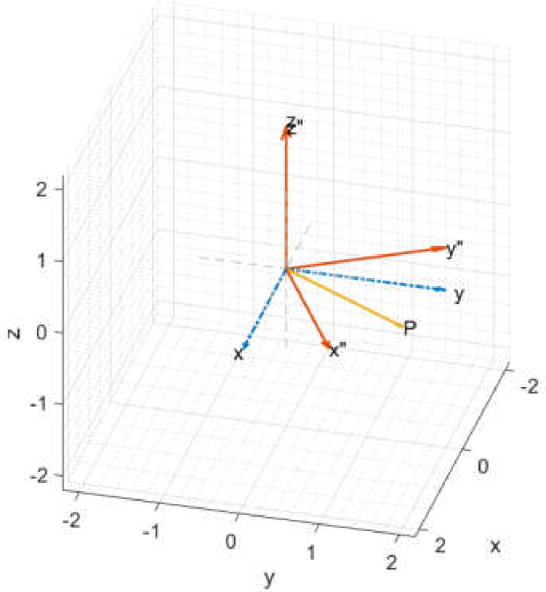

$${}^{A}P = {}^{A}_{B}R\,{}^{B}P$$

$${}^{A}P = \begin{bmatrix} 0.866 & -0.5 & 0 \\ 0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1.732 \\ 1 \\ 0 \end{bmatrix}$$

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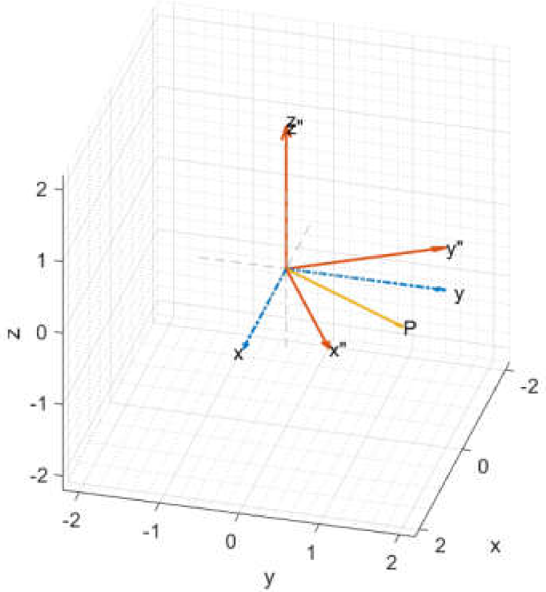

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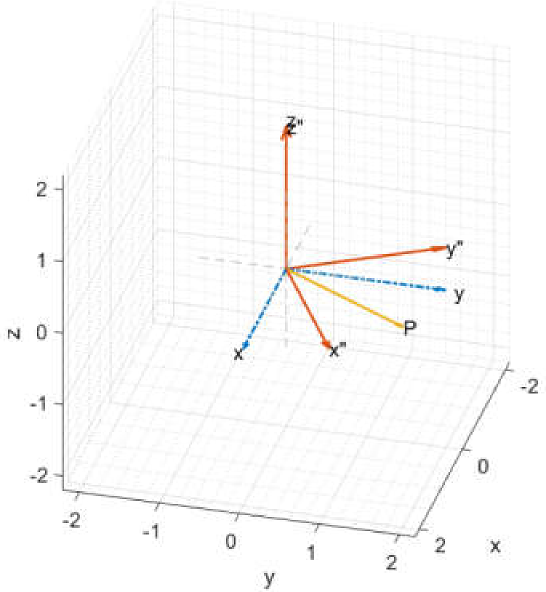

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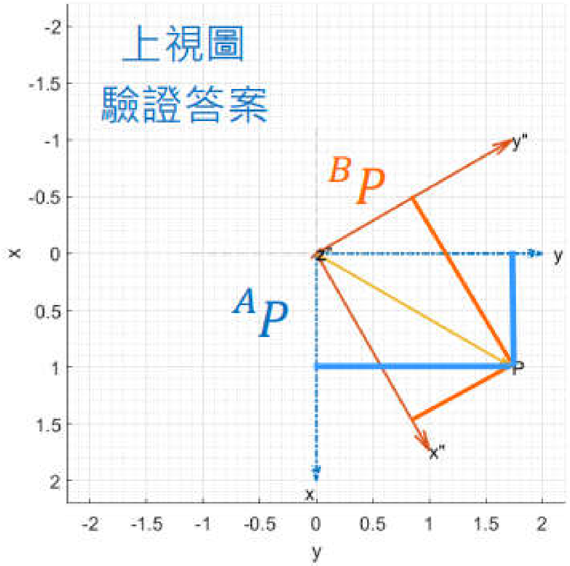

藍虛線: World Frame $\{A\}$

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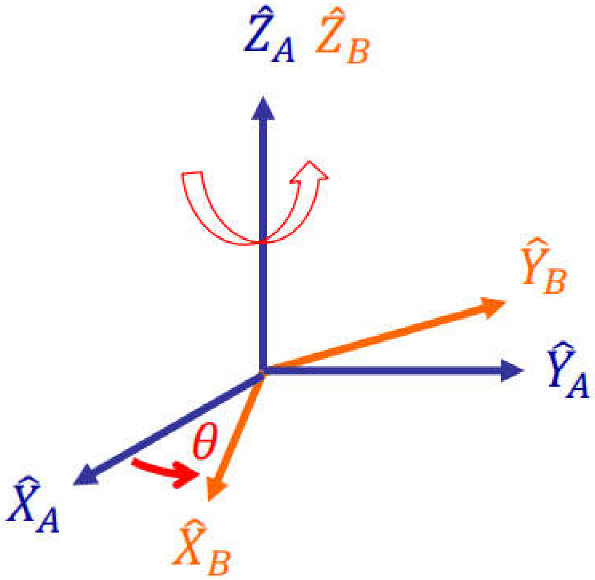

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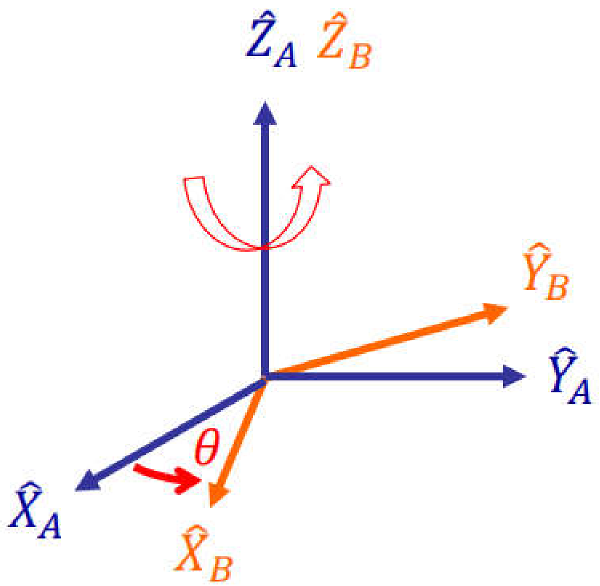

旋轉角度

$$R_{\hat{Z}_A}(\theta) = \begin{bmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

旋轉軸

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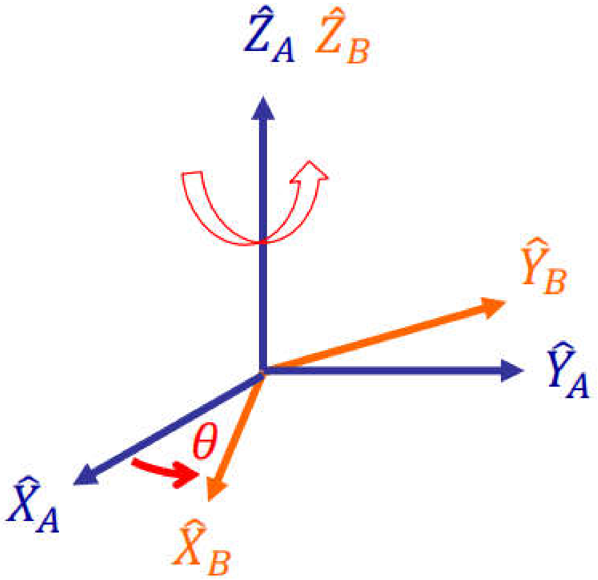

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旋轉軸

Note: $^A_BR = \begin{bmatrix} | & | & | \\ {}^A\hat{X}_B & {}^A\hat{Y}_B & {}^A\hat{Z}_B \\ | & | & | \end{bmatrix}$

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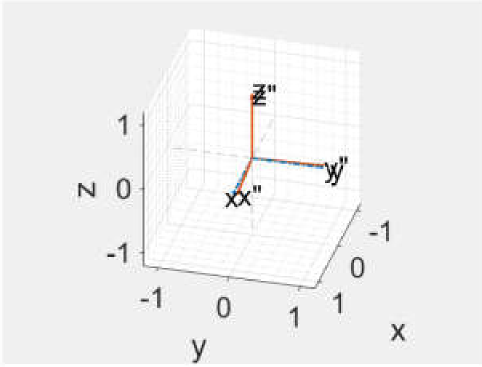

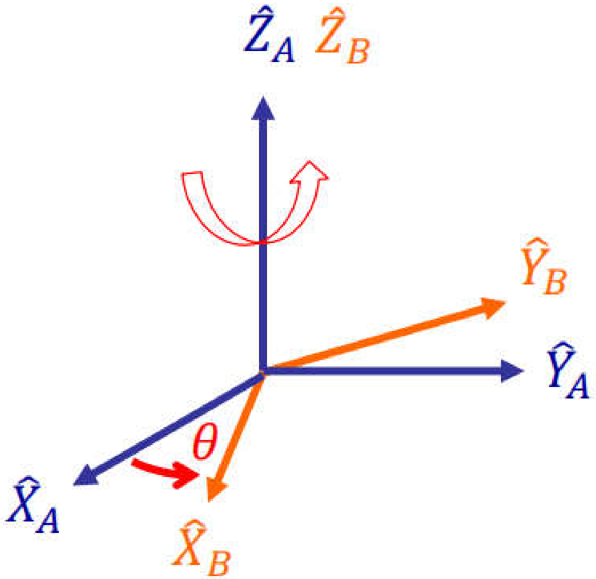

旋轉角度

$$R_{\hat{Z}_A}(\theta) = \begin{bmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

旋轉軸

$$= \begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note: $^A_BR = \begin{bmatrix} | & | & | \\ {}^A\hat{X}_B & {}^A\hat{Y}_B & {}^A\hat{Z}_B \\ | & | & | \end{bmatrix}$

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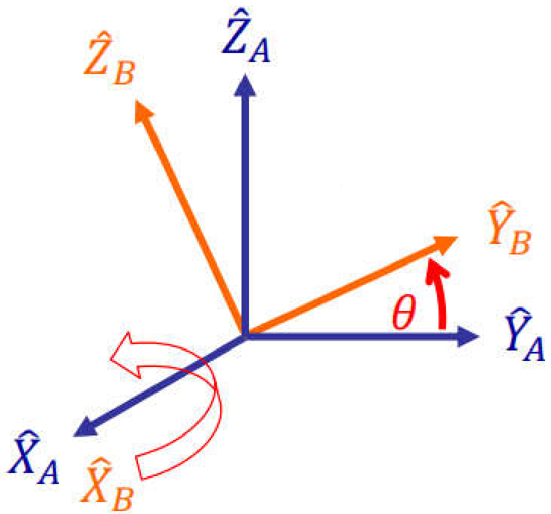
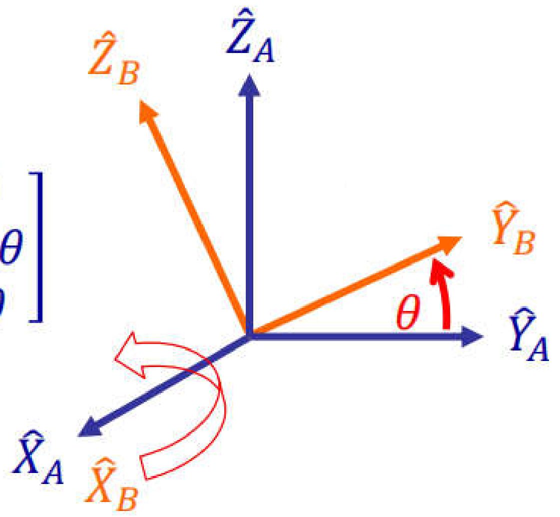


2.4 旋转矩阵

- About $\hat{X}_A$ with θ

$$R_{\hat{X}_A}(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\theta & -sin\theta \\ 0 & sin\theta & cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta & -s\theta \\ 0 & s\theta & c\theta \end{bmatrix}$$



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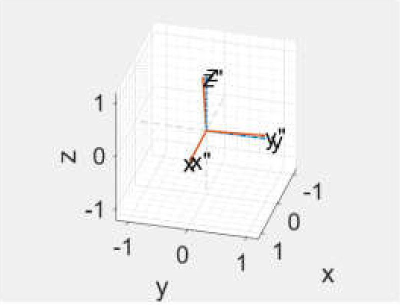

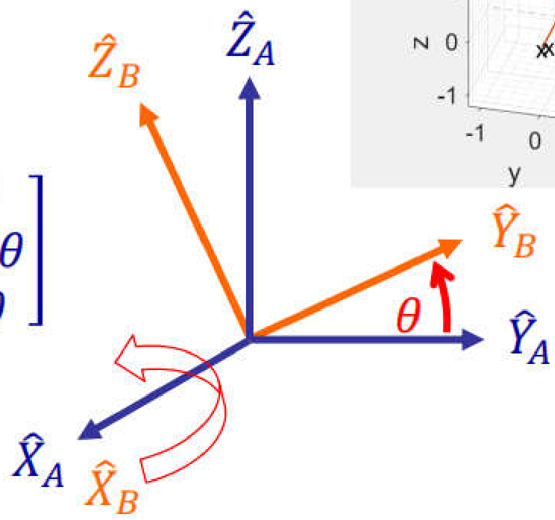

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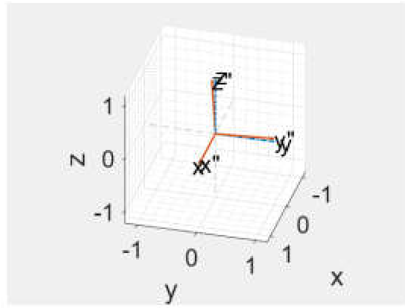
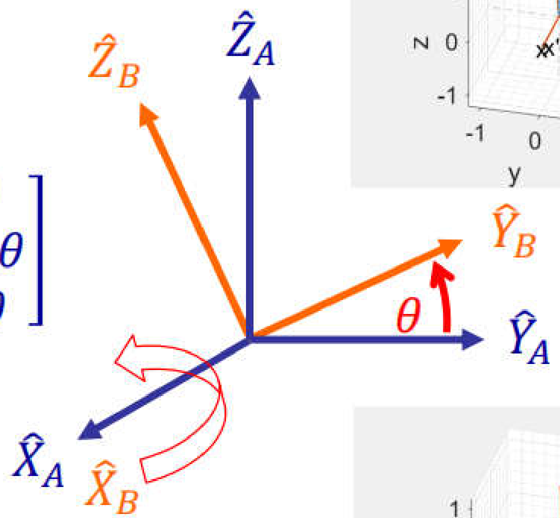
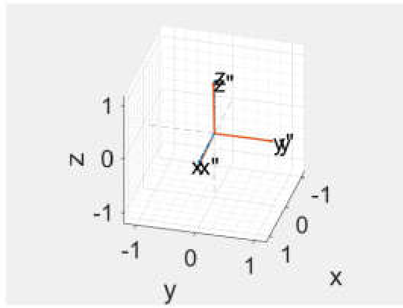
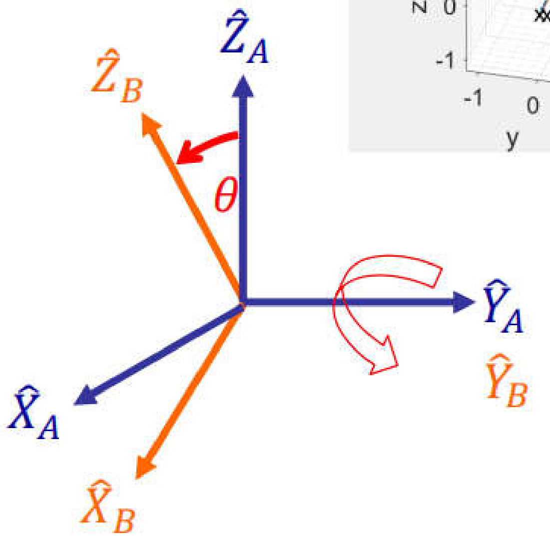




- About $\hat{Y}_A$ with θ

$$R_{\hat{Y}_A}(\theta)=$$

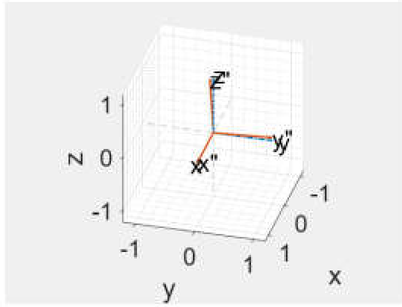


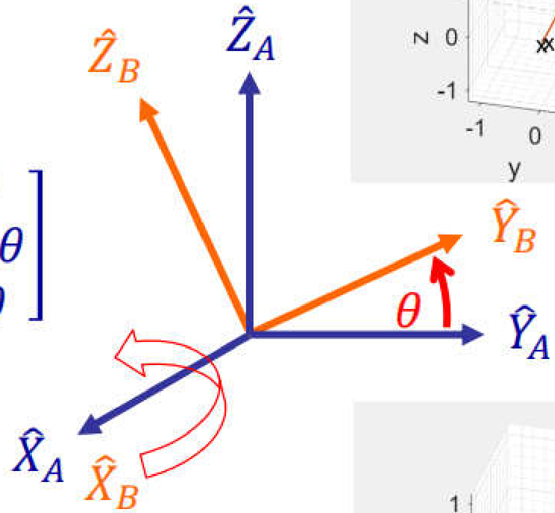


2.4 旋转矩阵

- About $\hat{X}_A$ with θ

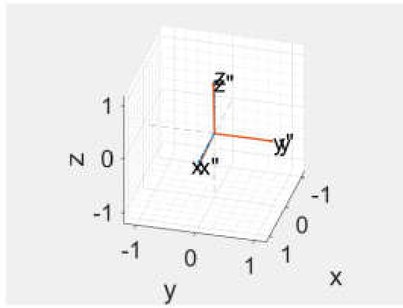
$$R_{\hat{X}_A}(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\theta & -sin\theta \\ 0 & sin\theta & cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta & -s\theta \\ 0 & s\theta & c\theta \end{bmatrix}$$

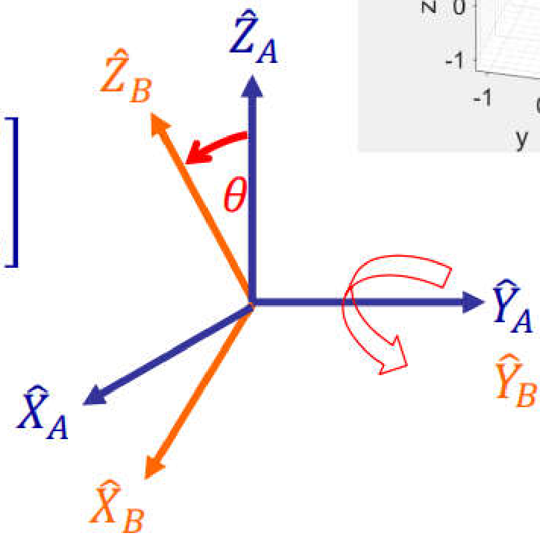




- About $\hat{Y}_A$ with θ

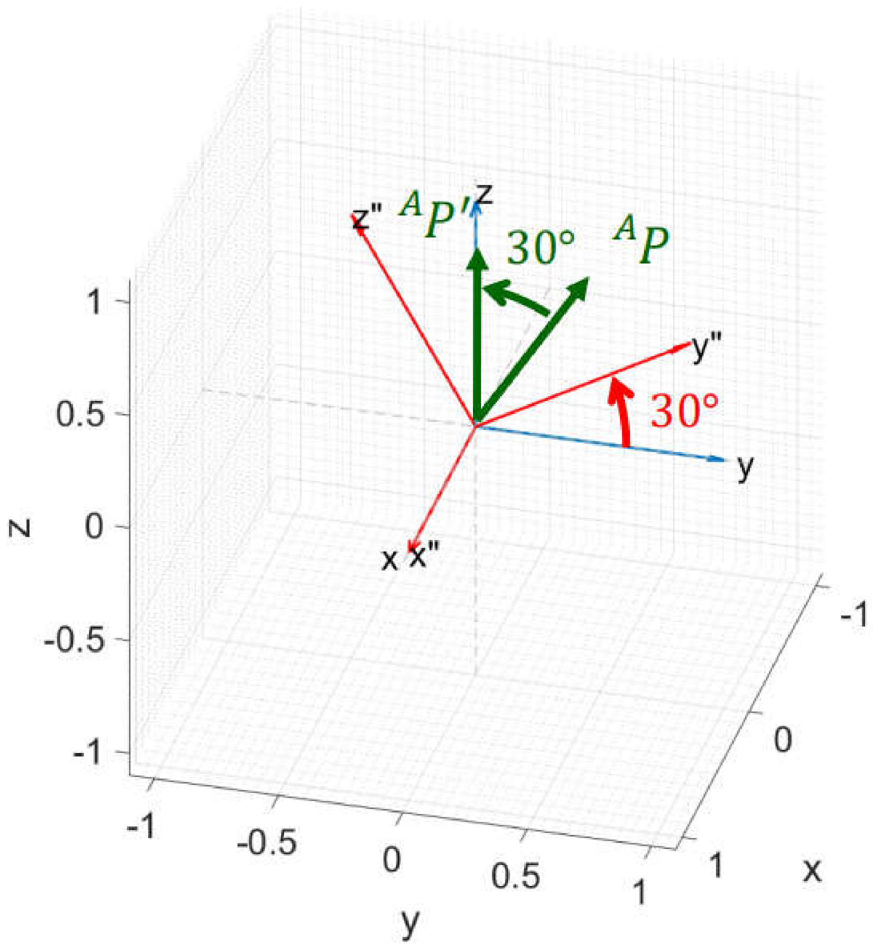
$$R_{\hat{Y}_A}(\theta) = \begin{bmatrix} cos\theta & 0 & sin\theta \\ 0 & 1 & 0 \\ -sin\theta & 0 & cos\theta \end{bmatrix} = \begin{bmatrix} c\theta & 0 & s\theta \\ 0 & 1 & 0 \\ -s\theta & 0 & c\theta \end{bmatrix}$$





2.4 旋转矩阵

- Ex: ${}^{A}P = [0 \quad 1 \quad 1.732]^T$ 對 $\hat{X}_A$ 軸旋轉30°，${}^{A}P' = ?$



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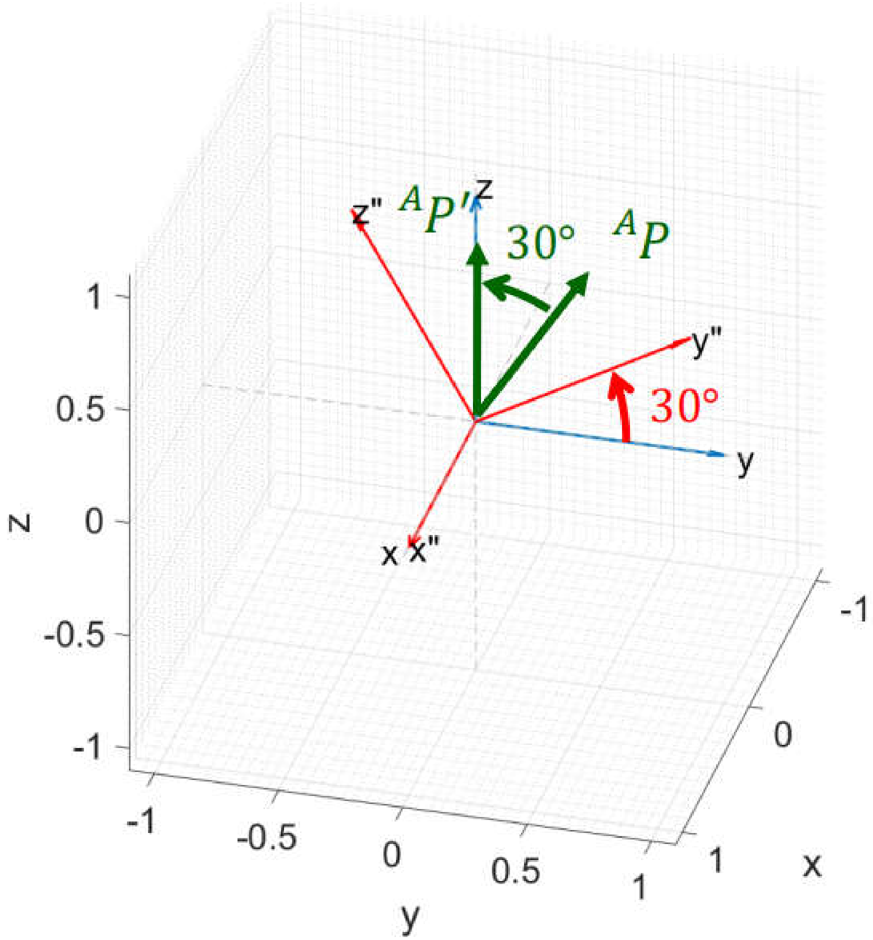

$$R_{\hat{X}_A}(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 30° & -\sin 30° \\ 0 & \sin 30° & \cos 30° \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.866 & -0.5 \\ 0 & 0.5 & 0.866 \end{bmatrix}$$

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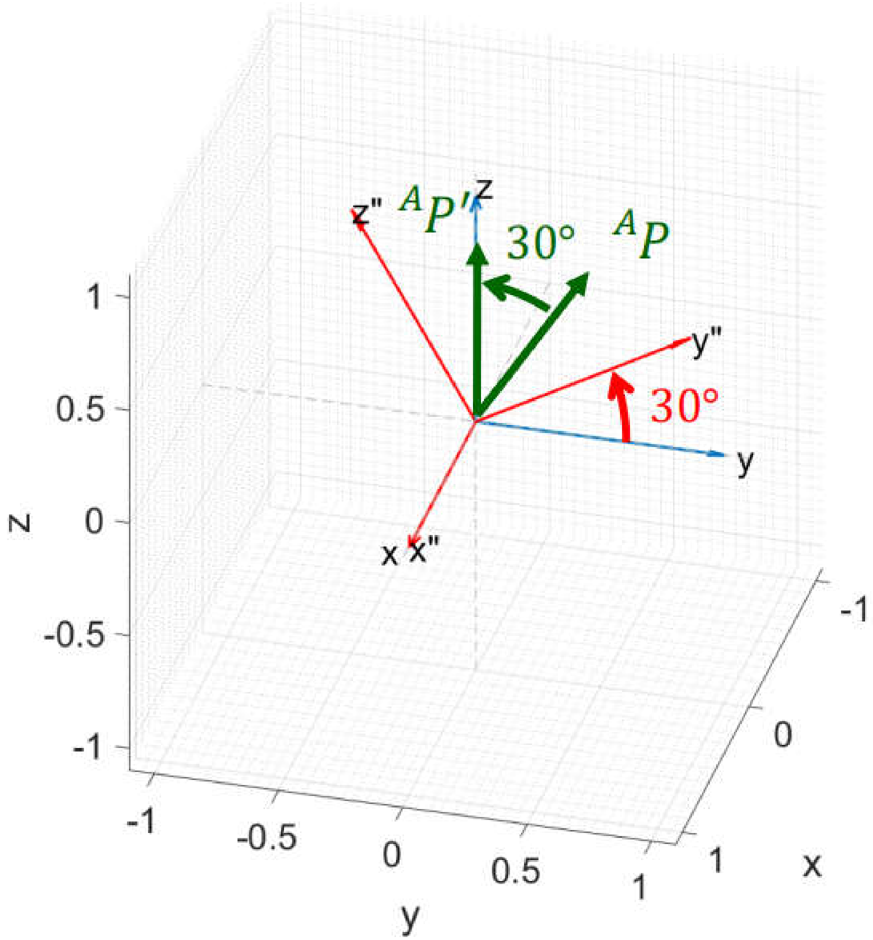

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$${}^{A}P' = R_{\hat{X}_A}(\theta)\, {}^{A}P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.866 & -0.5 \\ 0 & 0.5 & 0.866 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1.732 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

2.4 旋转矩阵

- Rotation matrix 的三種用法
  - 描述一個frame(相對於另一個frame)的姿態

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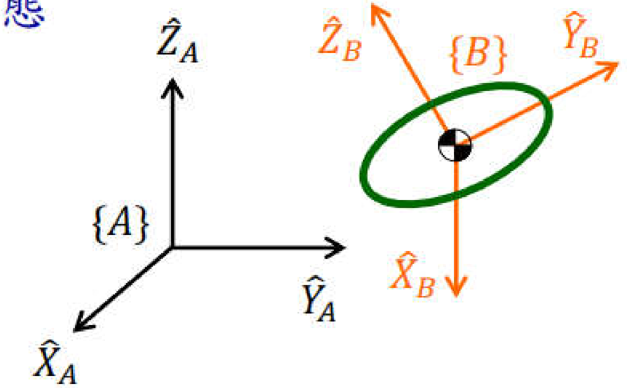

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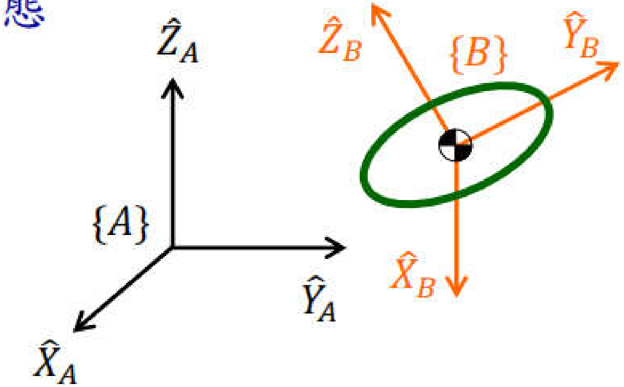
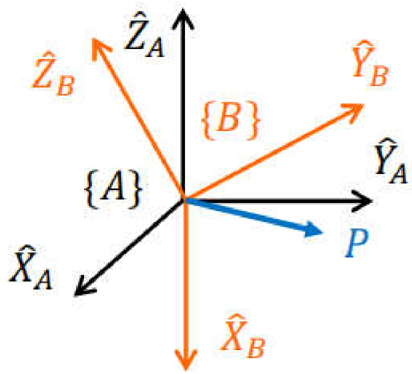


  - 將point由某一個frame的表達換到另一個和此frame僅有相對轉動的frame來表達

$$ {}^{A}P = {}^{A}_{B}R\,{}^{B}P $$



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2.4 旋转矩阵

Rotation matrix 的三種用法

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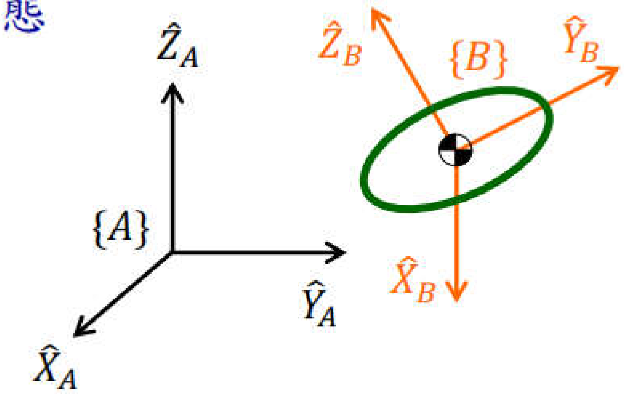

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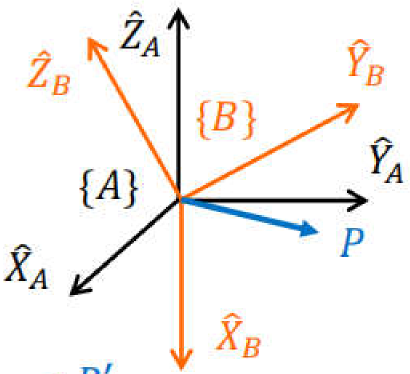
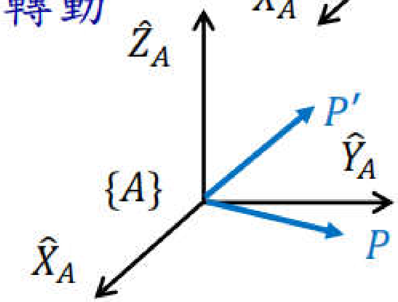


- 將point(vector)在同一個frame中進行轉動

$$ {}^{A}P' = R(\theta)\, {}^{A}P $$



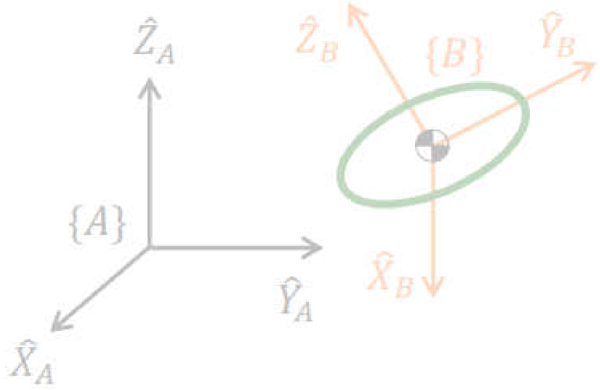
第二章 空间描述和变换

2.5 旋转矩阵与转角

❑ Rotation matrix 的三種用法

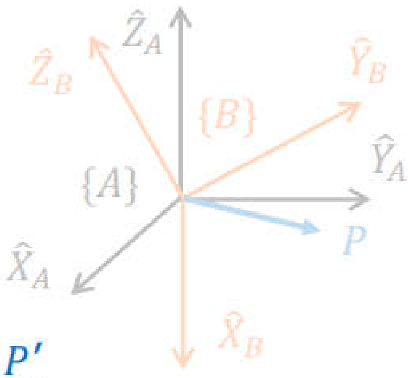
- 描述一個frame(相對於另一個frame)的姿態

$$ {}^A_BR = \begin{bmatrix} | & | & | \\ {}^A\hat{X}_B & {}^A\hat{Y}_B & {}^A\hat{Z}_B \\ | & | & | \end{bmatrix} $$



- 將point由某一個frame的表達換到另一個和此frame僅有相對轉動的frame來表達

$$ {}^AP = {}^A_BR\,{}^BP $$



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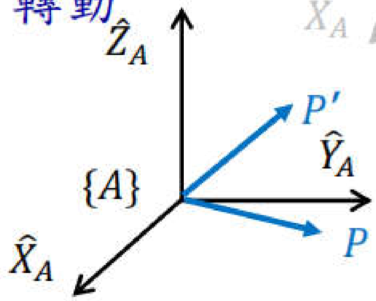


2.5 旋转矩阵与转角

- 空間中的Rotation是3 DOFs，那要如何把一般rotation matrix所表達的姿態，拆解成3次旋轉角度，以對應到3個DOFs？

2.5 旋转矩阵与转角

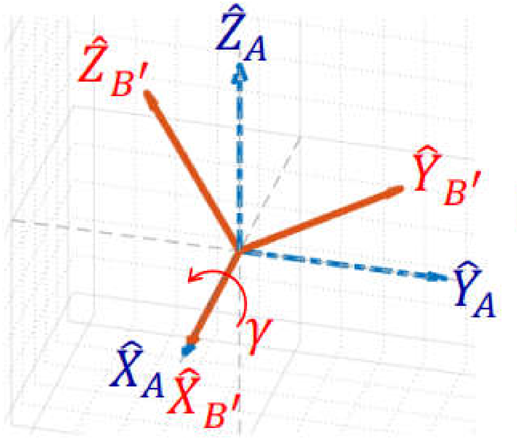
- 空間中的Rotation是3 DOFs，那要如何把一般rotation matrix所表達的姿態，拆解成3次旋轉角度，以對應到3個DOFs？

- 拆解成「三次旋轉連乘」所需注意事項
  - Rotation不是commutable，所以多次旋轉的先後順序需要明確定義
  - 旋轉轉軸也需要明確定義。是對「固定不動」的轉軸旋轉？或是對「轉動的frame當下所在」的轉軸旋轉？

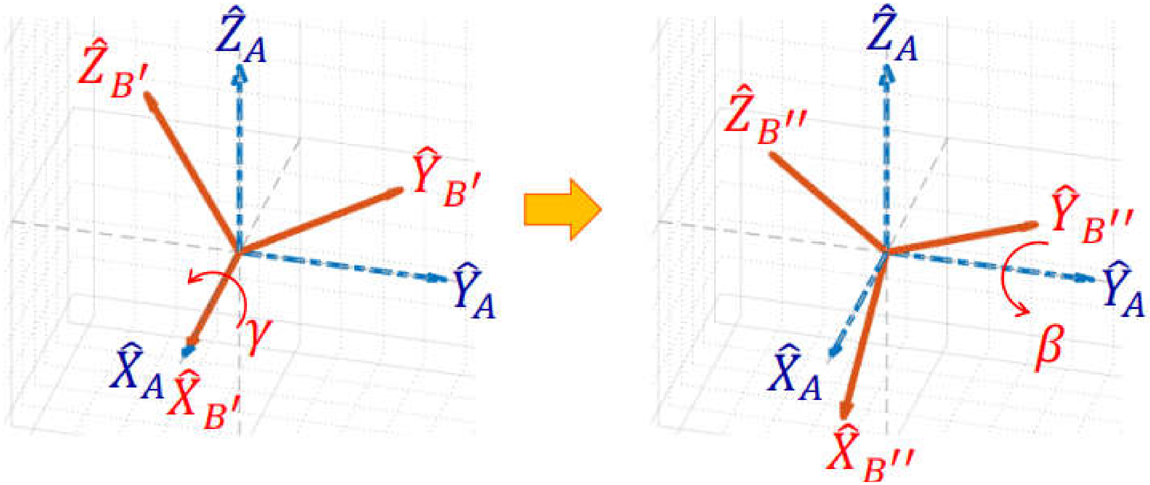
2.5 旋转矩阵与转角

- 空間中的Rotation是3 DOFs，那要如何把一般rotation matrix所表達的姿態，拆解成3次旋轉角度，以對應到3個DOFs？
- 拆解成「三次旋轉連乘」所需注意事項
  - Rotation不是commutable，所以多次旋轉的先後順序需要明確定義
  - 旋轉轉軸也需要明確定義。是對「固定不動」的轉軸旋轉？或是對「轉動的frame當下所在」的轉軸旋轉？
- 兩個拆解方式
  - 對方向「固定不動」的轉軸旋轉：Fixed angles
  - 對「轉動的frame當下所在」的轉軸方向旋轉：Euler angles

- X-Y-Z Fixed Angles – 由angles推算R

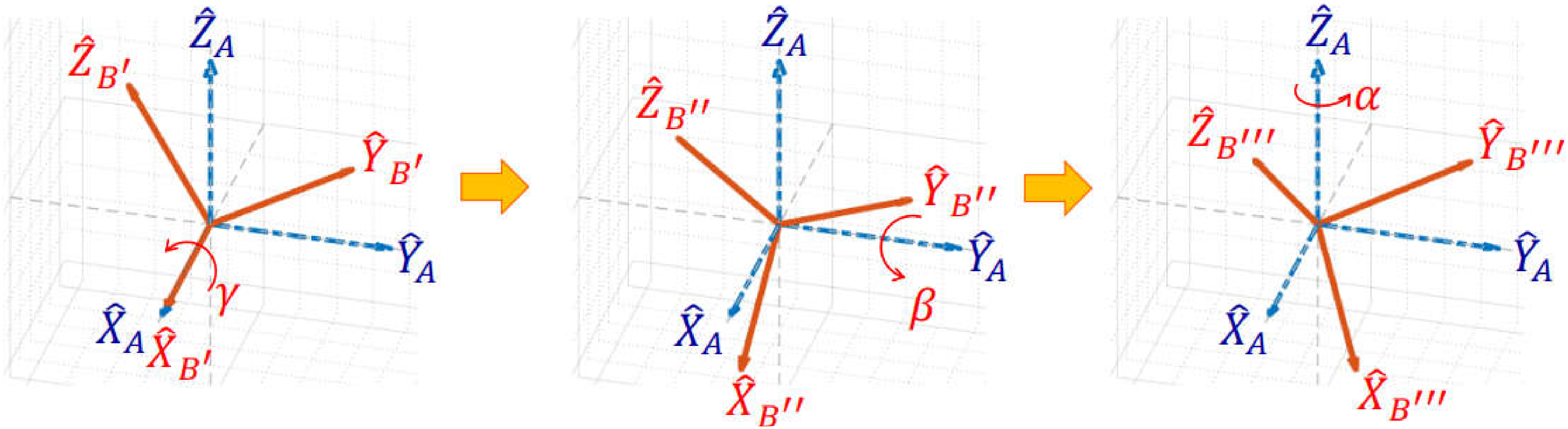


- X-Y-Z Fixed Angles – 由angles推算R

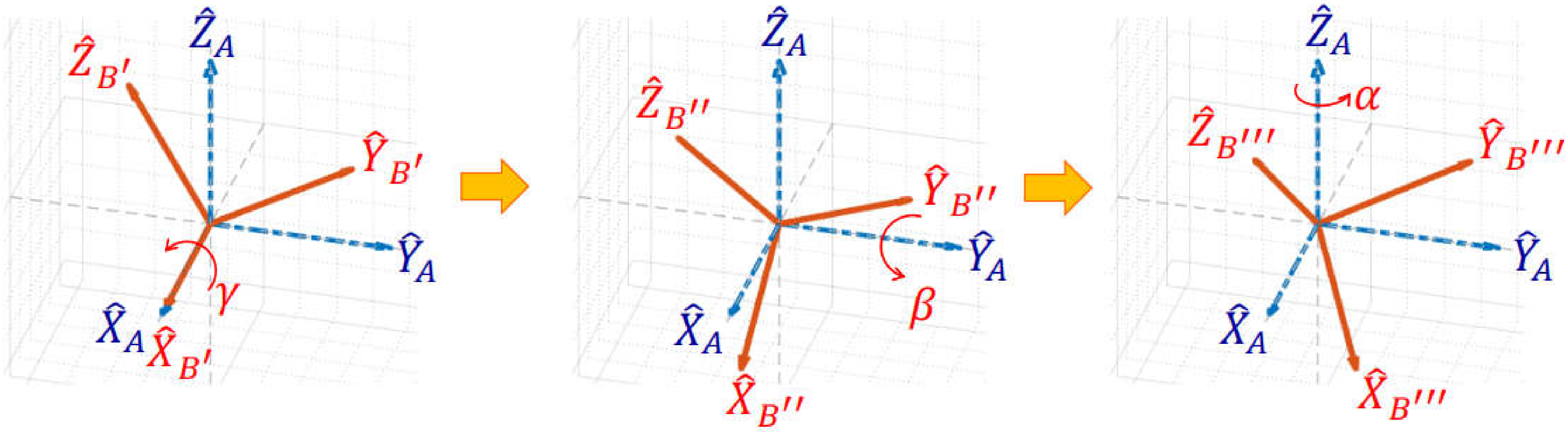


2.5 旋转矩阵与转角

- X-Y-Z Fixed Angles – 由angles推算R



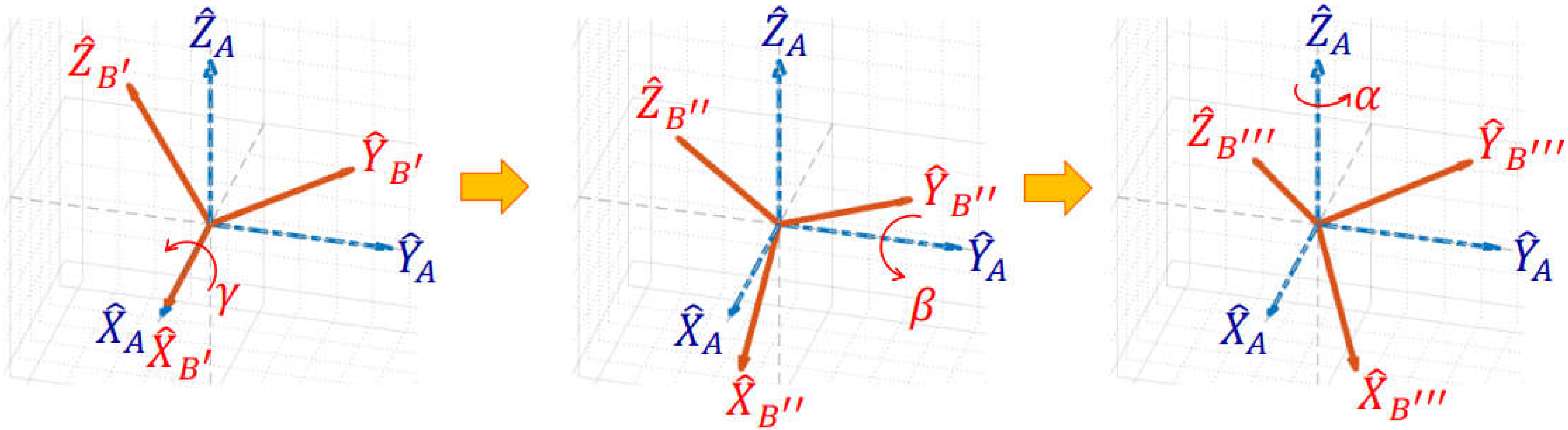
- X-Y-Z Fixed Angles – 由angles推算R



$${}^{A}_{B}R_{XYZ}(\gamma,\beta,\alpha)=R_Z(\alpha)R_Y(\beta)R_X(\gamma)$$

2.5 旋转矩阵与转角

- X-Y-Z Fixed Angles – 由angles推算R

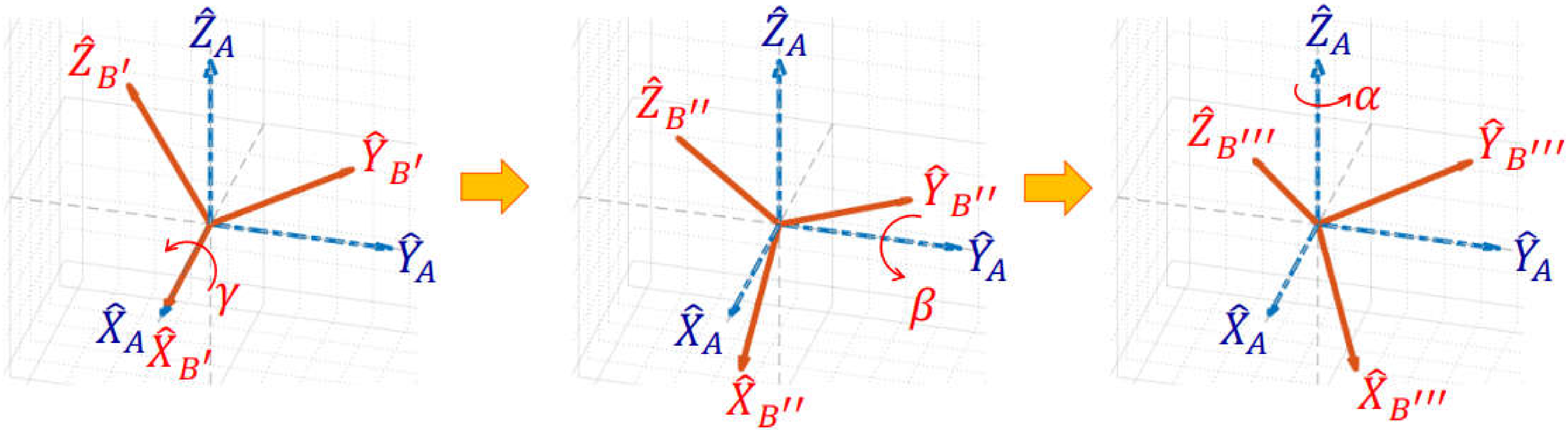


$${}^{A}_{B}R_{XYZ}(\gamma,\beta,\alpha) = R_Z(\alpha)R_Y(\beta)R_X(\gamma) \quad v' = {}^{A}_{B}Rv = R_3R_2R_1v$$

先轉的放「後面」：以operator來想，對某一個向量，「以同一個座標為基準」，進行轉動或移動的操作

2.5 旋转矩阵与转角

- X-Y-Z Fixed Angles – 由angles推算R



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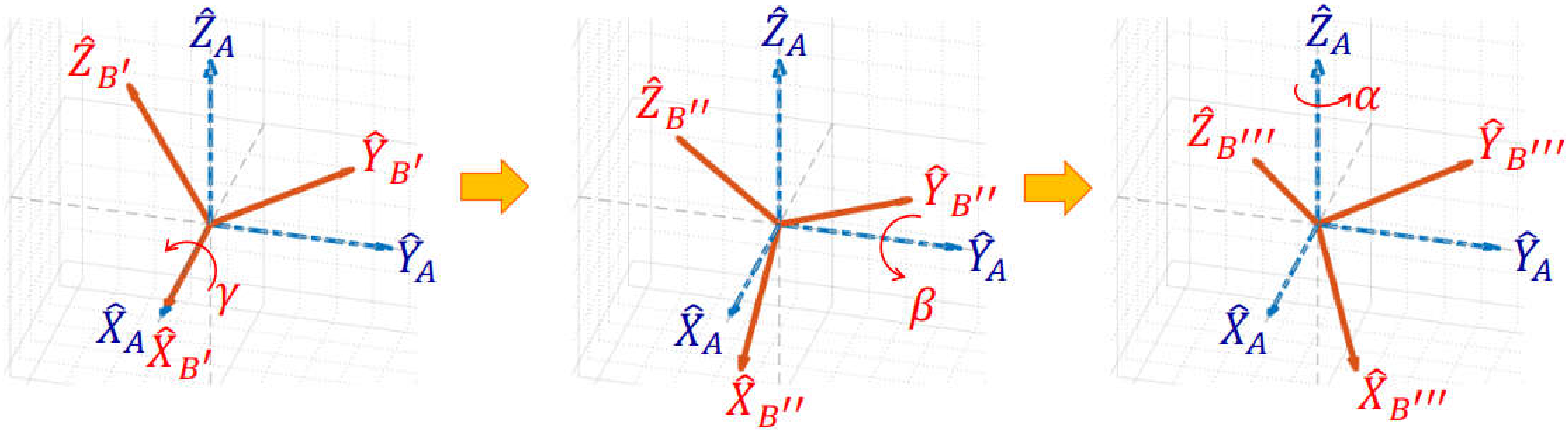
先轉的放「後面」：以operator來想，對某一個向量，「以同一個座標為基準」，進行轉動或移動的操作

$$= \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix}$$

2.5 旋转矩阵与转角

- X-Y-Z Fixed Angles – 由angles推算R



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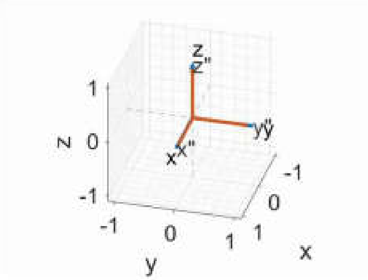
$$= \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix}$$

2.5 旋转矩阵与转角

- Ex: 以Fixed Angles旋轉：「先對X軸旋轉60度，後對Y軸旋轉30度」和「先對Y軸旋轉30度，後對X軸旋轉60度」各自的 ${}^{A}_{B}R$ 分別是?

2.5 旋转矩阵与转角

- Ex: 以Fixed Angles旋轉：「先對X軸旋轉60度，後對Y軸旋轉30度」和「先對Y軸旋轉30度，後對X軸旋轉60度」各自的 ${}^{A}_{B}R$ 分別是?

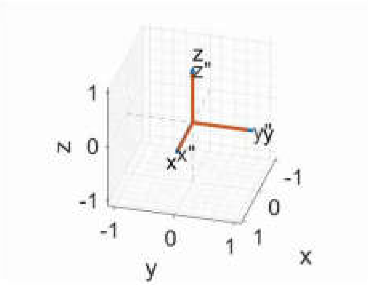


先對X轉60度，再對Y轉30度

$$
{}^{A}_{B}R_{XYZ}(\gamma,\beta,\alpha)=R_Z(0)R_Y(30)R_X(60)=\begin{bmatrix}0.866 & 0.433 & 0.25\\ 0 & 0.5 & -0.866\\ -0.5 & 0.75 & 0.433\end{bmatrix}
$$

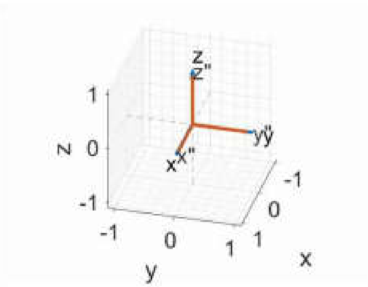
2.5 旋转矩阵与转角

- Ex: 以Fixed Angles旋轉：「先對X軸旋轉60度，後對Y軸旋轉30度」和「先對Y軸旋轉30度，後對X軸旋轉60度」各自的 ${}^{A}_{B}R$ 分別是?



先對X轉60度，再對Y轉30度

$$ {}^{A}_{B}R_{XYZ}(\gamma,\beta,\alpha)=R_Z(0)R_Y(30)R_X(60)=\begin{bmatrix}0.866 & 0.433 & 0.25\\ 0 & 0.5 & -0.866\\ -0.5 & 0.75 & 0.433\end{bmatrix} $$

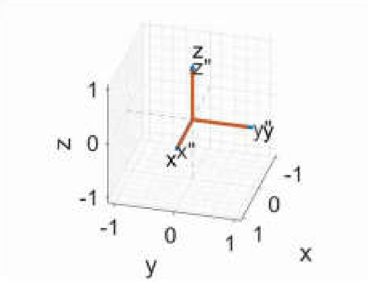


先對Y轉30度，再對X轉60度

$$ {}^{A}_{B}R_{XYZ}(\gamma,\beta,\alpha)=R_Z(0)R_X(60)R_Y(30)=\begin{bmatrix}0.866 & 0 & 0.5\\ 0.433 & 0.5 & -0.75\\ -0.25 & 0.866 & 0.433\end{bmatrix} $$

2.5 旋转矩阵与转角

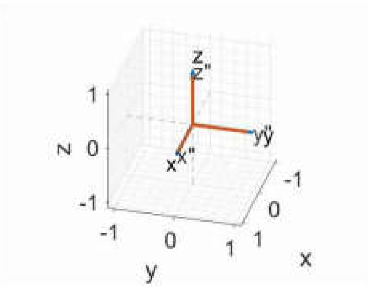
- Ex: 以Fixed Angles旋轉：「先對X軸旋轉60度，後對Y軸旋轉30度」和「先對Y軸旋轉30度，後對X軸旋轉60度」各自的 ${}^{A}_{B}R$ 分別是?



先對X轉60度，再對Y轉30度

$$ {}^{A}_{B}R_{XYZ}(\gamma,\beta,\alpha) = R_Z(0)R_Y(30)R_X(60) = \begin{bmatrix} 0.866 & 0.433 & 0.25 \\ 0 & 0.5 & -0.866 \\ -0.5 & 0.75 & 0.433 \end{bmatrix} $$

$$\neq$$



先對Y轉30度，再對X轉60度

$$ {}^{A}_{B}R_{XYZ}(\gamma,\beta,\alpha) = R_Z(0)R_X(60)R_Y(30) = \begin{bmatrix} 0.866 & 0 & 0.5 \\ 0.433 & 0.5 & -0.75 \\ -0.25 & 0.866 & 0.433 \end{bmatrix} $$

2.5 旋转矩阵与转角

- X-Y-Z Fixed Angles – 由R推算angles

$$
{}^{A}_{B}R_{XYZ}(\gamma,\beta,\alpha)=\begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix}=\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}
$$

2.5 旋转矩阵与转角

- X-Y-Z Fixed Angles – 由R推算angles

$$
{}^{A}_{B}R_{XYZ}(\gamma,\beta,\alpha)=\begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix}=\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}
$$

If $\beta \neq 90°$

$$\beta = Atan2(-r_{31}, \sqrt{{r_{11}}^2 + {r_{21}}^2})$$

$$\alpha = Atan2(r_{21}/c\beta, r_{11}/c\beta)$$

$$\gamma = Atan2(r_{32}/c\beta, r_{33}/c\beta)$$

$-90° \leq \beta \leq 90°$

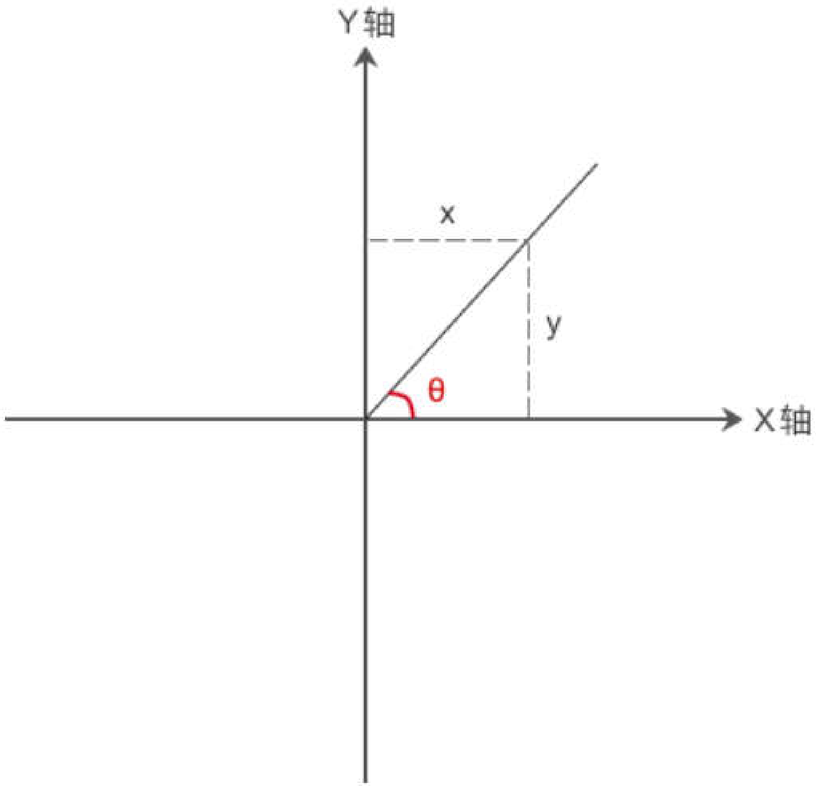
Single solution

2.5 旋转矩阵与转角

atan和atan2都是反正切函数，如：有两个点 point(x1,y1), 和 point(x2,y2);
那么这两个点形成的斜率的弧度计算方法分别是:
float radian = atan((y2-y1)/(x2-x1));
或float radian = atan2(y2-y1, x2-x1);

atan 和 atan2 区别在于:
1. 参数的填写方式不同;
2. atan的取值范围为（-pi/2, pi/2），atan2的取值范围为(-pi, pi];
3. atan2 的优点在于x2-x1等于0时依然可以计算，但是atan函数除零会出错;

2.5 旋转矩阵与转角



$$\operatorname{atan2}(y,x)=\begin{cases}\arctan(\frac{y}{x}) & x>0\\ \pi+\arctan(\frac{y}{x}) & y\geq 0, x<0\\ -\pi+\arctan(\frac{y}{x}) & y<0, x<0\\ \frac{\pi}{2} & y>0, x=0\\ -\frac{\pi}{2} & y<0, x=0\end{cases}$$

2.5 旋转矩阵与转角

- X-Y-Z Fixed Angles – 由R推算angles

$$ {}^{A}_{B}R_{XYZ}(\gamma,\beta,\alpha) = \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} $$

If $\beta \neq 90°$

$$ \beta = Atan2(-r_{31}, \sqrt{{r_{11}}^2 + {r_{21}}^2}) $$
$$ \alpha = Atan2(r_{21}/c\beta, r_{11}/c\beta) $$
$$ \gamma = Atan2(r_{32}/c\beta, r_{33}/c\beta) $$

$-90° \leq \beta \leq 90°$

Single solution

If $\beta = 90°$

$$ \alpha = 0° $$
$$ \gamma = Atan2(r_{12}, r_{22}) $$

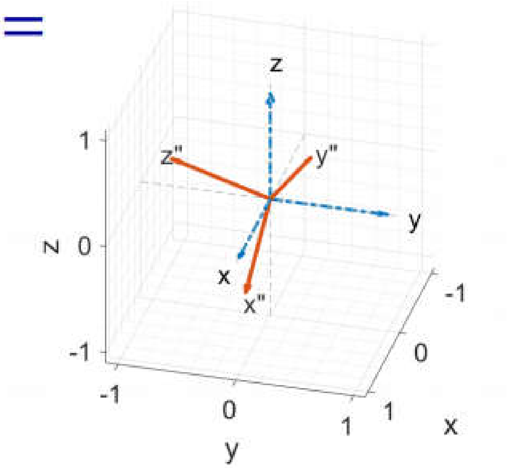
If $\beta = -90°$

$$ \alpha = 0° $$
$$ \gamma = -Atan2(r_{12}, r_{22}) $$

2.5 旋转矩阵与转角

- Ex: 以X-Y-Z Fixed Angles方法，反算$R=$

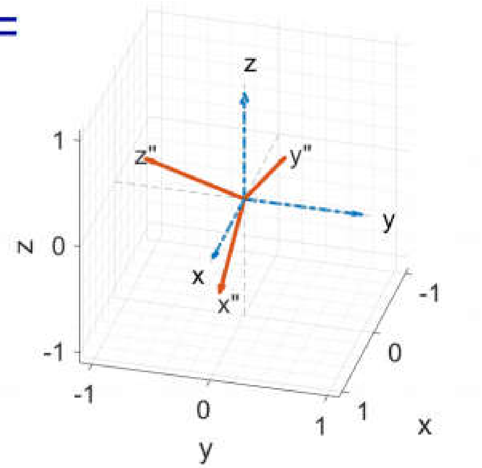
$$\begin{bmatrix} 0.866 & 0.433 & 0.25 \\ 0 & 0.5 & -0.866 \\ -0.5 & 0.75 & 0.433 \end{bmatrix}$$的angles



2.5 旋转矩阵与转角

- Ex: 以X-Y-Z Fixed Angles方法，反算$R =$

$$\begin{bmatrix} 0.866 & 0.433 & 0.25 \\ 0 & 0.5 & -0.866 \\ -0.5 & 0.75 & 0.433 \end{bmatrix}$$ 的angles



$$\beta = Atan2\left(-r_{31}, \sqrt{{r_{11}}^2 + {r_{21}}^2}\right) = Atan2\left(-(-0.5), \sqrt{0.866^2 + 0^2}\right) = 30°$$

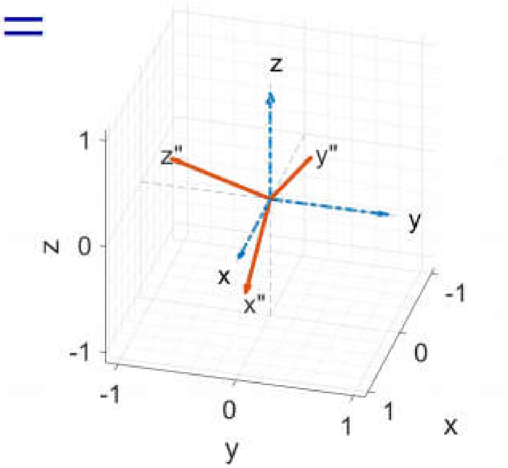
$$\alpha = Atan2\left(\frac{r_{21}}{c\beta}, \frac{r_{11}}{c\beta}\right) = Atan2\left(\frac{0}{\cos 30}, \frac{0.866}{\cos 30}\right) = 0°$$

$$\gamma = Atan2\left(\frac{r_{32}}{c\beta}, \frac{r_{33}}{c\beta}\right) = Atan2\left(\frac{0.75}{\cos 30}, \frac{0.433}{\cos 30}\right) = 60°$$

2.5 旋转矩阵与转角

- Ex: 以X-Y-Z Fixed Angles方法，反算$R=$

$$\begin{bmatrix} 0.866 & 0.433 & 0.25 \\ 0 & 0.5 & -0.866 \\ -0.5 & 0.75 & 0.433 \end{bmatrix}$$ 的angles



$$\beta = Atan2\left(-r_{31}, \sqrt{{r_{11}}^2 + {r_{21}}^2}\right) = Atan2\left(-(-0.5), \sqrt{0.866^2 + 0^2}\right) = 30°$$

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$$\gamma = Atan2\left(\frac{r_{32}}{c\beta}, \frac{r_{33}}{c\beta}\right) = Atan2\left(\frac{0.75}{\cos 30}, \frac{0.433}{\cos 30}\right) = 60°$$

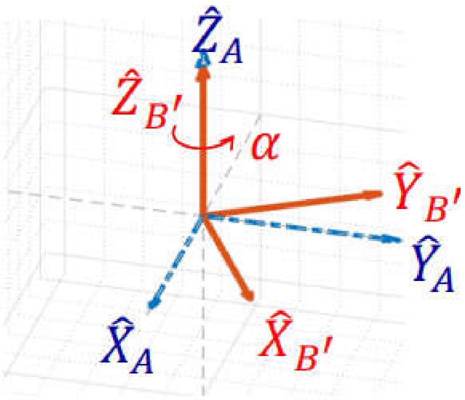
➡ $\boldsymbol{R_Z(0)R_Y(30)R_X(60)}$

先對X轉60度，再對Y轉30度

和Fixed Angles -2的結果相同

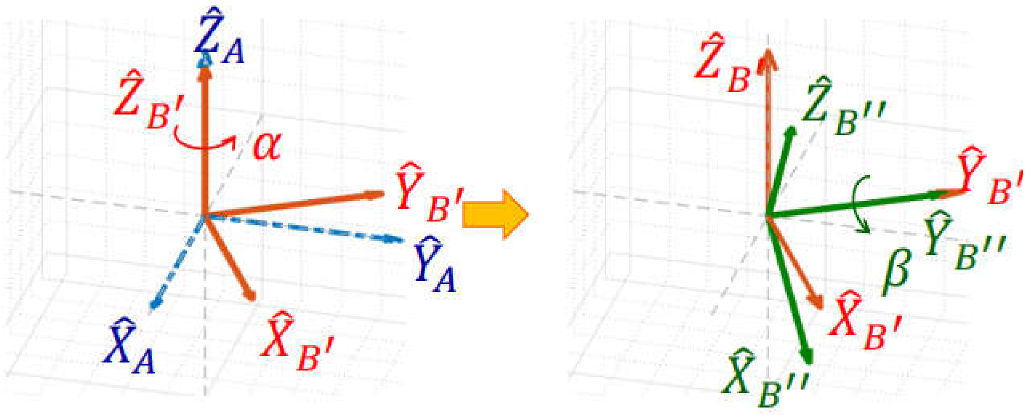
2.5 旋转矩阵与转角

- Z-Y-X Euler Angles - 由angles推算R



2.5 旋转矩阵与转角

- Z-Y-X Euler Angles - 由angles推算R



2.5 旋转矩阵与转角

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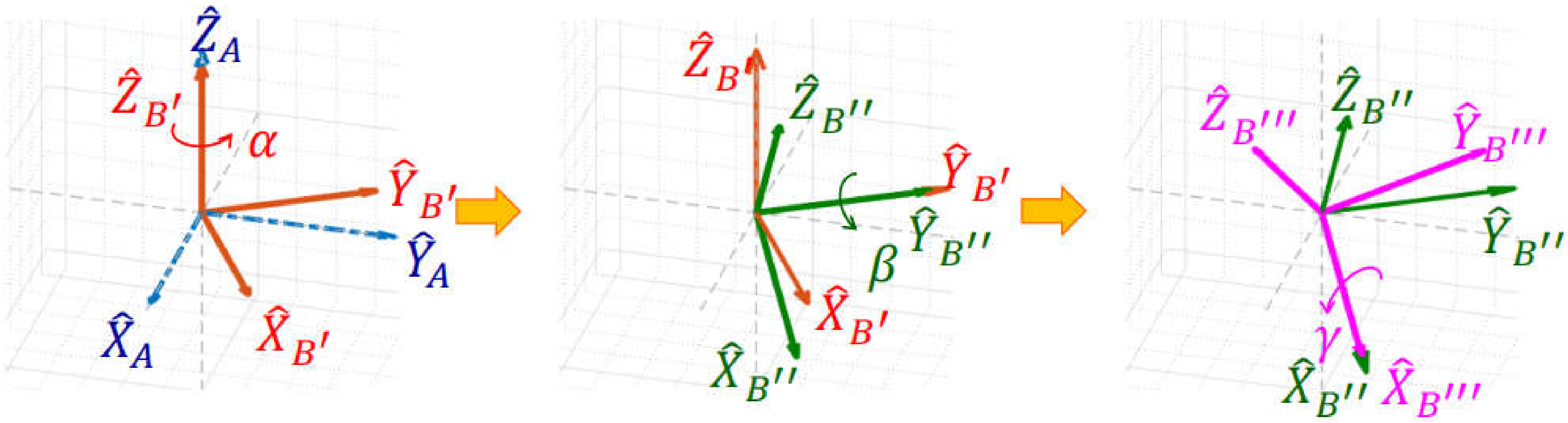
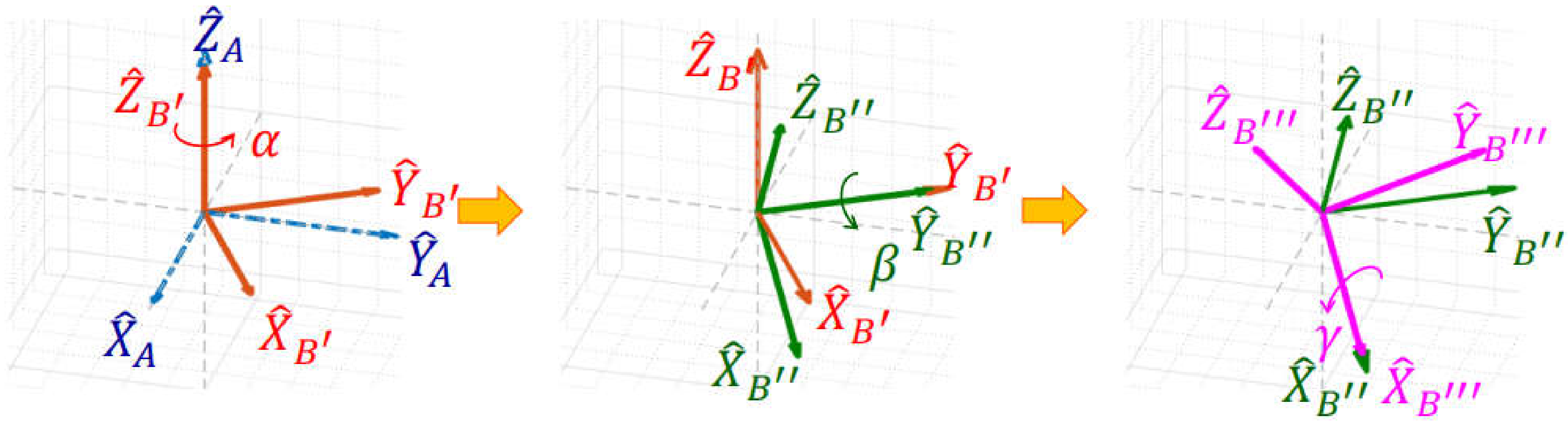


2.5 旋转矩阵与转角

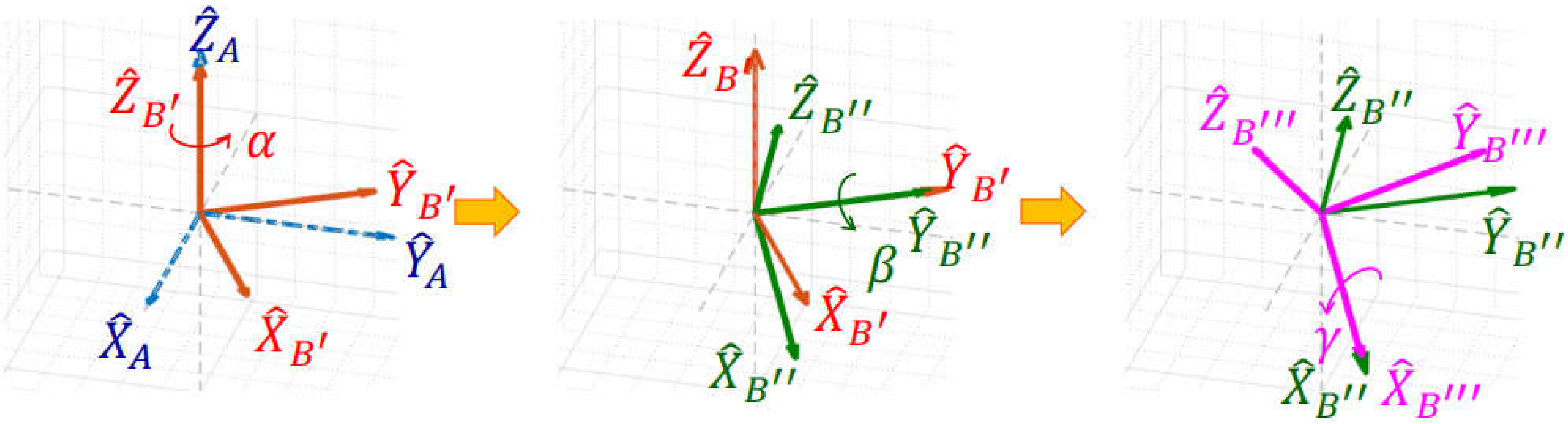
- Z-Y-X Euler Angles - 由angles推算R



$$ {}^{A}_{B}R_{Z'Y'X'}(\alpha,\beta,\gamma) = {}^{A}_{B'}R\,{}^{B'}_{B''}R\,{}^{B''}_{B}R = R_{Z'}(\alpha)R_{Y'}(\beta)R_{X'}(\gamma) $$

2.5 旋转矩阵与转角

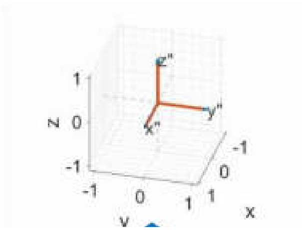
- Z-Y-X Euler Angles - 由angles推算R



$$ {}^A_BR_{Z'Y'X'}(\alpha,\beta,\gamma) = {}^A_{B'}R\,{}^{B'}_{B''}R\,{}^{B''}_{B}R = R_{Z'}(\alpha)R_{Y'}(\beta)R_{X'}(\gamma) $$

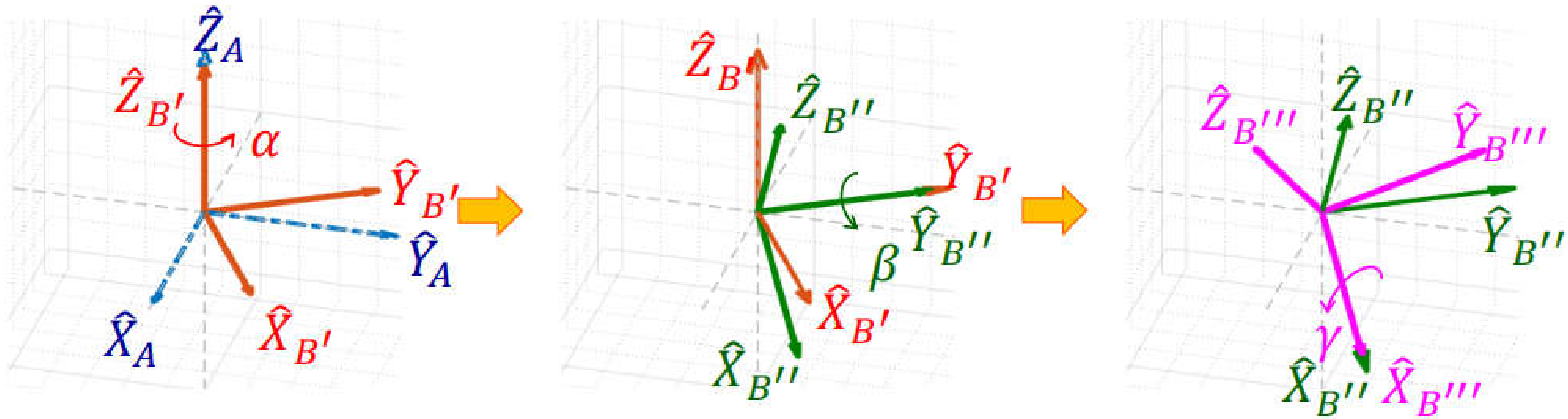
先轉的放「前面」：以mapping來想，對某一個向量，從最後一個frame「逐漸轉動或移動」來回到第一個frame

$$ {}^AP = {}^A_BR\,{}^BP = R_1R_2R_3\,{}^BP $$



2.5 旋转矩阵与转角

- Z-Y-X Euler Angles - 由angles推算R



$$ {}^A_BR_{Z'Y'X'}(\alpha,\beta,\gamma) = {}^A_{B'}R\,{}^{B'}_{B''}R\,{}^{B''}_{B}R = R_{Z'}(\alpha)R_{Y'}(\beta)R_{X'}(\gamma) $$

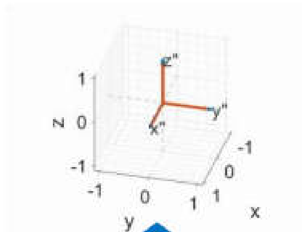
先轉的放「前面」：以mapping來想，對某一個向量，從最後一個frame「逐漸轉動或移動」來回到第一個frame

$$ {}^AP = {}^A_BR\,{}^BP = R_1R_2R_3\,{}^BP $$

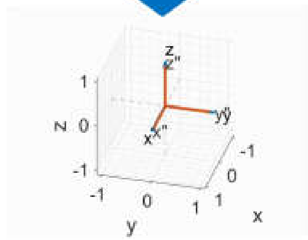
$$ = \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix} $$

$$ = R_Z(\alpha)R_Y(\beta)R_X(\gamma) = {}^A_BR_{XYZ}(\gamma,\beta,\alpha) $$

和X-Y-Z Fixed angle得到一樣的R



最後得出相同的R



2.5 旋转矩阵与转角

- Ex: 以Euler Angles旋轉：「先對X軸旋轉60度，後對Y軸旋轉30度」和「先對Y軸旋轉30度，後對X軸旋轉60度」各自的 ${}^{A}_{B}R$ 分別是?

2.5 旋转矩阵与转角

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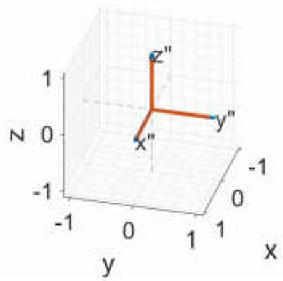
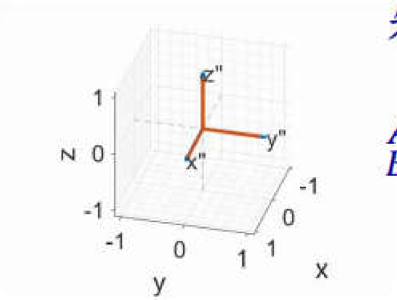


先對X轉60度，再對Y轉30度

$$
{}^{A}_{B}R_{X'Y'Z'}(\gamma,\beta,\alpha)=R_{X'}(60)R_{Y'}(30)=\begin{bmatrix}0.866 & 0 & 0.5\\ 0.433 & 0.5 & -0.75\\ -0.25 & 0.866 & 0.433\end{bmatrix}
$$

2.5 旋转矩阵与转角

- Ex: 以Euler Angles旋轉：「先對X軸旋轉60度，後對Y軸旋轉30度」和「先對Y軸旋轉30度，後對X軸旋轉60度」各自的 ${}^A_BR$ 分別是?



先對X轉60度，再對Y轉30度

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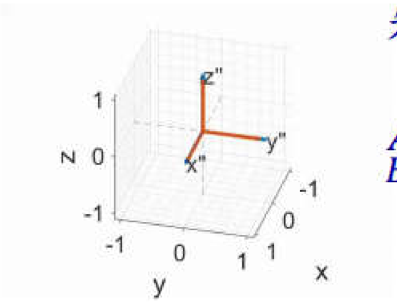

先對Y轉30度，再對X轉60度

$$ {}^A_BR_{X'Y'Z'}(\gamma,\beta,\alpha) = R_{Y'}(30)R_{X'}(60) = \begin{bmatrix} 0.866 & 0.433 & 0.25 \\ 0 & 0.5 & -0.866 \\ -0.5 & 0.75 & 0.433 \end{bmatrix} $$

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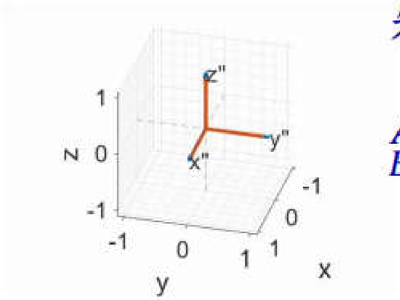

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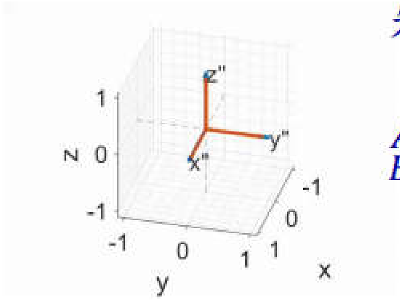
$$\neq$$



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- Ex: Euler(Y30, X60) v.s. Fixed(X60, Y30)

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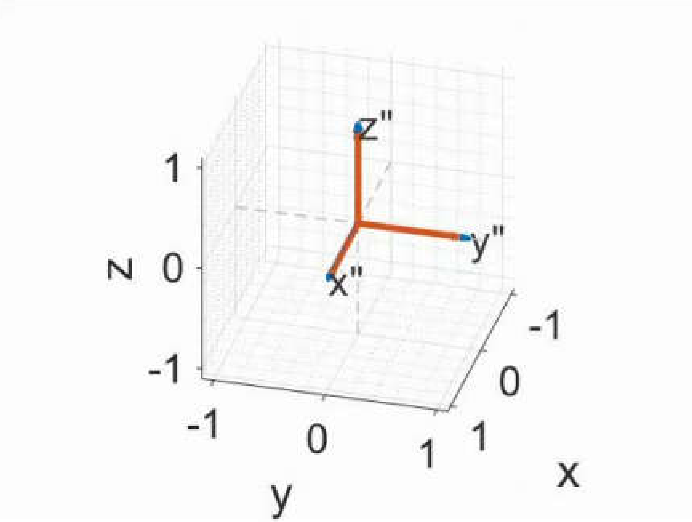

Euler Angles:

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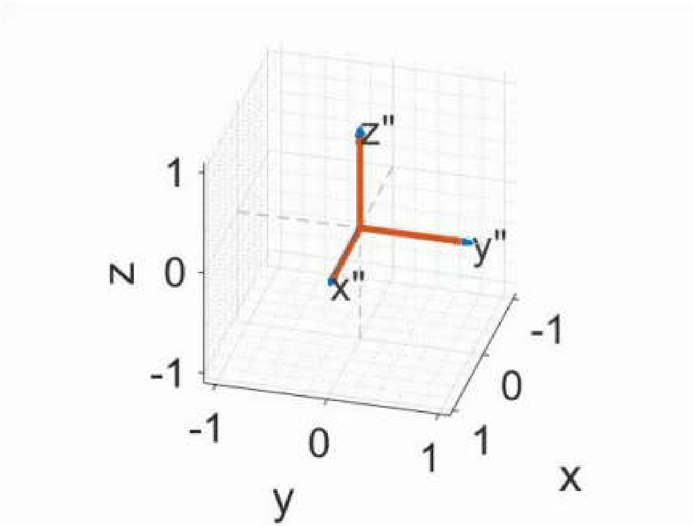

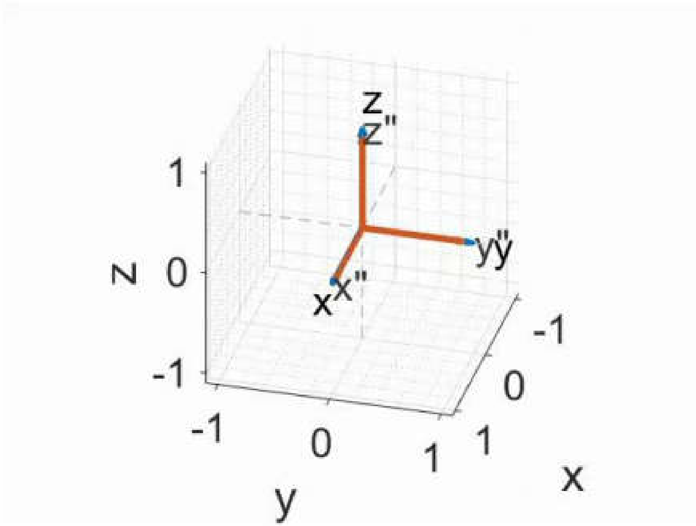

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$$

Fixed Angles:

先對X轉60度，再對Y轉30度

$$
{}^{A}_{B}R_{XYZ}(\gamma,\beta,\alpha) = R_{Y}(30)R_{X}(60) = \begin{bmatrix} 0.866 & 0.433 & 0.25 \\ 0 & 0.5 & -0.866 \\ -0.5 & 0.75 & 0.433 \end{bmatrix}
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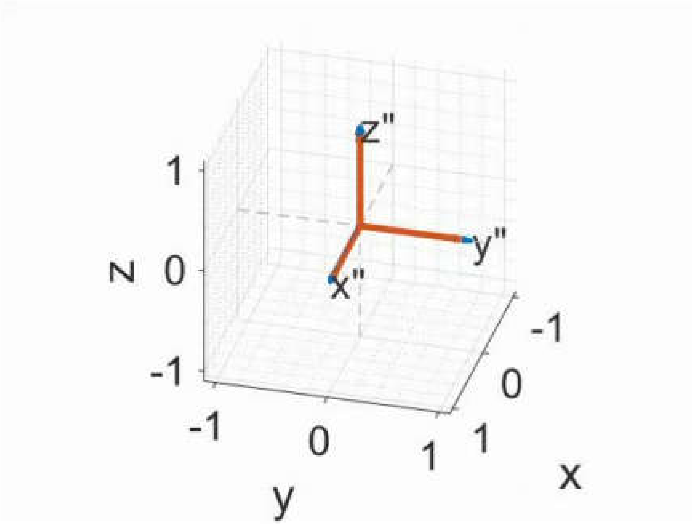

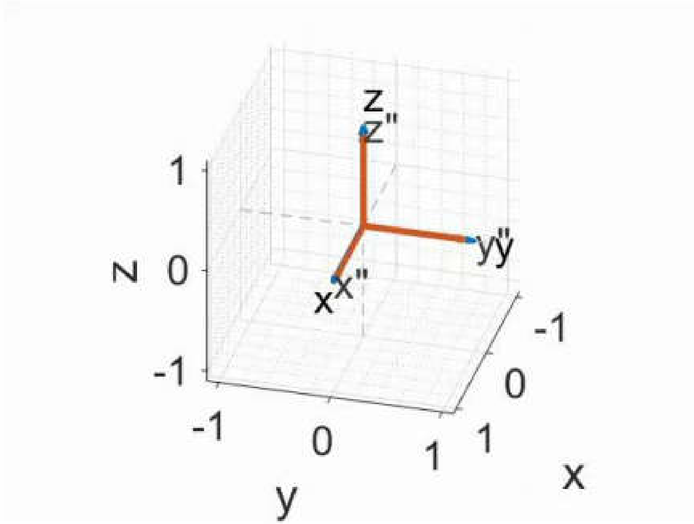

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$$

=

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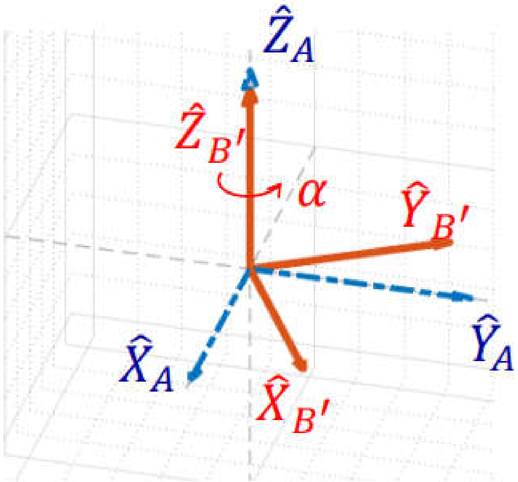
- Z-Y-Z Euler Angles - 由angles推算R



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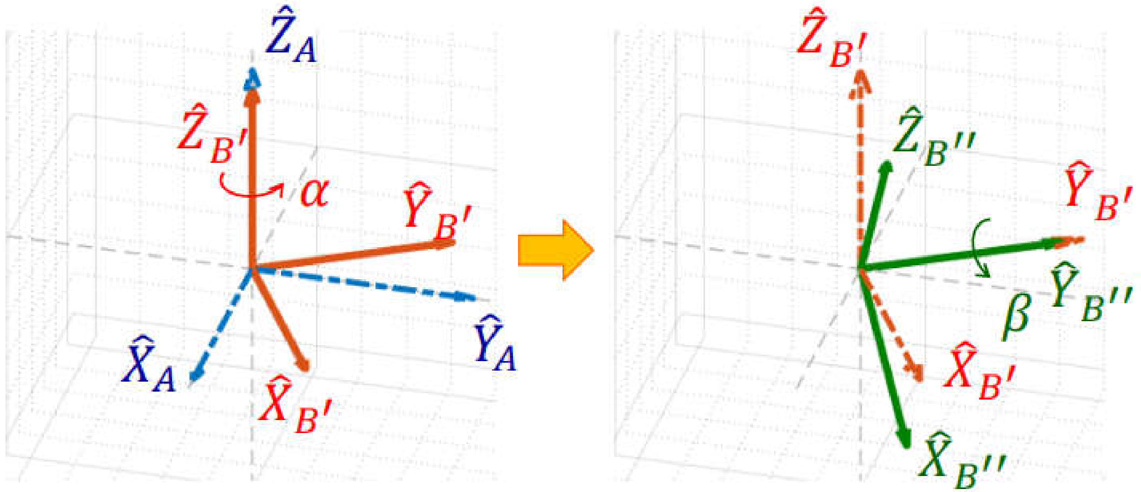

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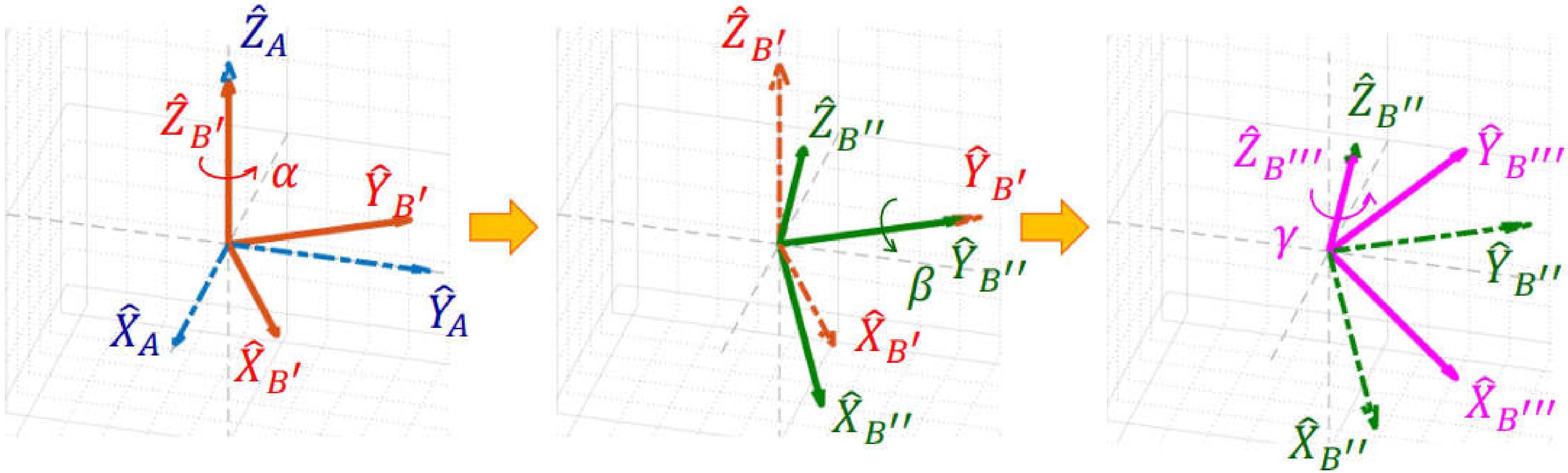

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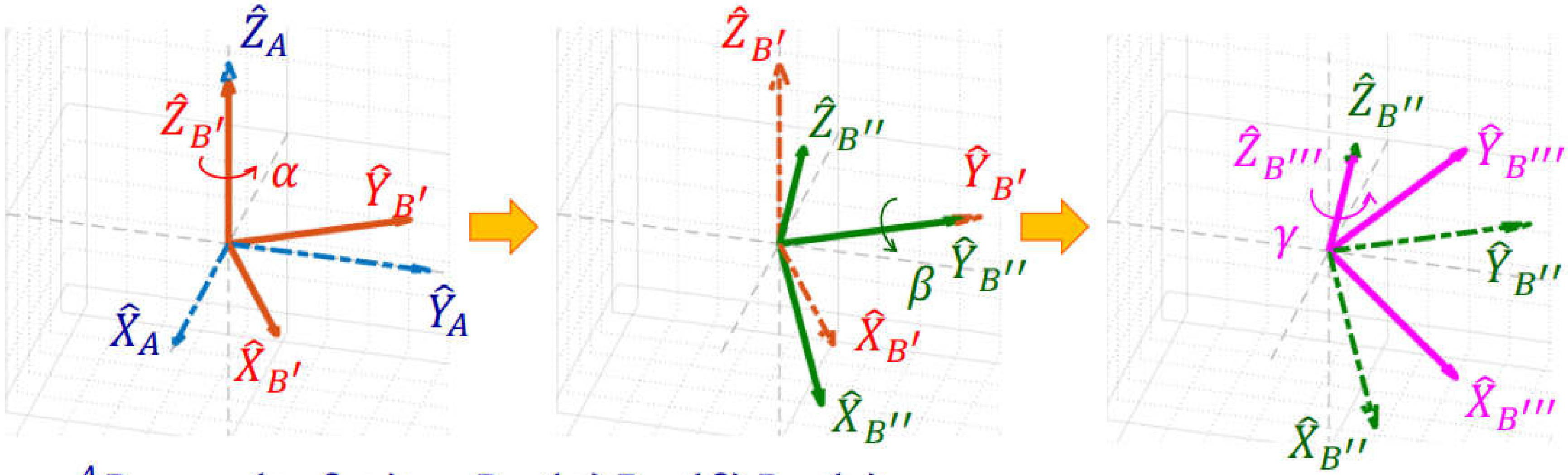
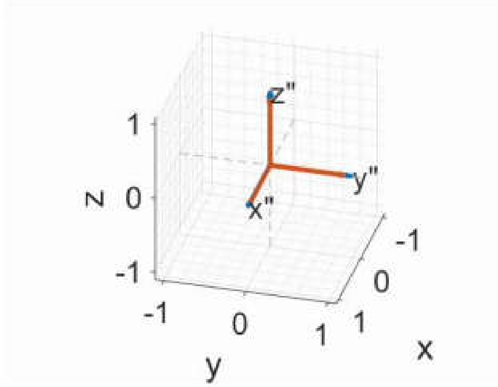


$${}^{A}_{B}R_{Z'Y'Z'}(\alpha,\beta,\gamma) = R_{Z'}(\alpha)R_{Y'}(\beta)R_{Z'}(\gamma)$$

先轉的放「前面」

$$= \begin{bmatrix} c\alpha c\beta c\gamma - s\alpha s\gamma & -c\alpha c\beta s\gamma - s\alpha c\gamma & c\alpha s\beta \\ s\alpha c\beta c\gamma + c\alpha s\gamma & -s\alpha c\beta s\gamma + c\alpha c\gamma & s\alpha s\beta \\ -s\beta c\gamma & s\beta s\gamma & c\beta \end{bmatrix}$$



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$$

If $\beta \neq 0°$

$$\beta = Atan2(\sqrt{r_{31}{}^2 + r_{32}{}^2}, r_{33})$$

$$\alpha = Atan2(r_{23}/s\beta, r_{13}/s\beta)$$

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If $\beta = 0°$

$\alpha = 0°$

$\gamma = Atan2(-r_{12}, r_{11})$

If $\beta = 180°$

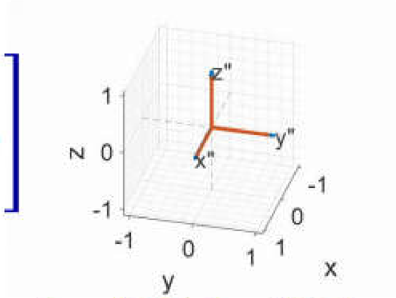
$\alpha = 0°$

$\gamma = Atan2(r_{12}, -r_{11})$

2.5 旋转矩阵与转角

- Ex: Revisit Euler Angles-2的範例

$$ {}^{A}_{B}R_{X'Y'Z'}(60,30,0) = R_{X'}(60)R_{Y'}(30) = \begin{bmatrix} 0.866 & 0 & 0.5 \\ 0.433 & 0.5 & -0.75 \\ -0.25 & 0.866 & 0.433 \end{bmatrix} $$



$R_{X'}(60)R_{Y'}(30)$

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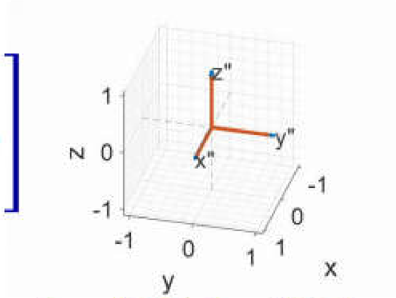

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- 若以ZYZ的公式反算，Euler Angles 為何？

$$ \beta = Atan2\left(\sqrt{{r_{31}}^2 + {r_{32}}^2}, r_{33}\right) = Atan2\left(\sqrt{(-0.25)^2 + 0.866^2}, 0.433\right) = 64.3° $$

$$ \alpha = Atan2\left(\frac{r_{23}}{s\beta}, \frac{r_{13}}{s\beta}\right) = Atan2\left(\frac{-0.75}{s\beta}, \frac{0.5}{s\beta}\right) = -56.3° $$

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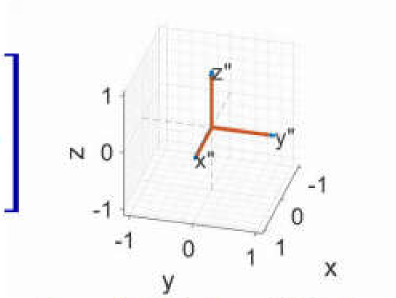

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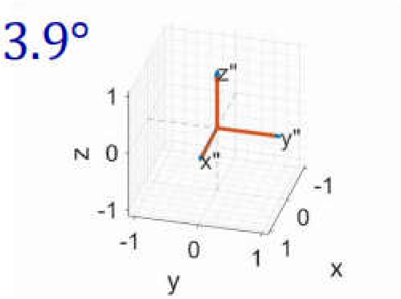
➡ $\boldsymbol{R_{Z'}(-56.3)R_{Y'}(64.3)R_{Z'}(73.9)}$

先對Z轉 -56.3°,對Y轉64.3°,最後對Z轉73.9°



$R_{Z'}(-56.3)R_{Y'}(64.3)R_{Z'}(73.9)$

2.5 旋转矩阵与转角

- Euler/Fixed angles
  - 12種 Euler angles 和 12種 fixed angles
  - 存在Duality – 共12種對principal axes連三次轉動的拆解方法

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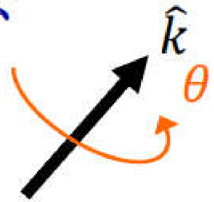
- Angle-axis表達法

對$\hat{k}$旋轉θ

unit vector



Unit vector裡2個參數，轉角1個參數，也為3 DOFs

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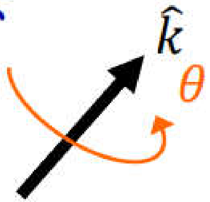


Unit vector裡2個參數，轉角1個參數，也為3 DOFs

- Quaternion表達法

$$q = \epsilon_4 + \epsilon_1\hat{\imath}+\epsilon_2\hat{\jmath}+\epsilon_3\hat{k}$$
$$= cos\frac{\theta}{2} + sin\frac{\theta}{2}(k_x\hat{\imath} + k_y\hat{\jmath} + k_z\hat{k})$$

note $\epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2 + \epsilon_4^2 = 1$

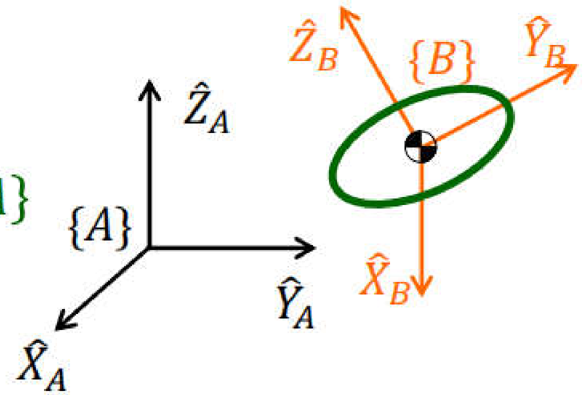
4個參數+1個限制條件，也為3 DOFs

第二章 空间描述和变换

2.6 齐次变换矩阵

- 「導讀-3」的問題：該如何整合表達剛體的狀態？
- ⇨在剛體(Rigid body)上建立frame，常建立在質心上
  - 移動：由body frame 的原點位置判定

$$ {}^{A}P_{B\,org} = \begin{bmatrix} P_{B\,x} \\ P_{B\,y} \\ P_{B\,z} \end{bmatrix} = \text{origin of } \{B\} \text{ represented in } \{A\} $$



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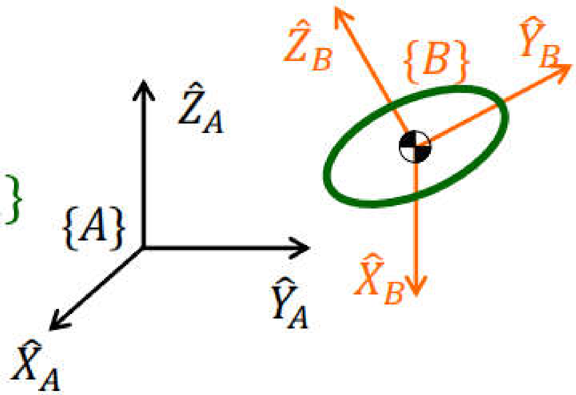

  - 轉動：由body frame的姿態判定

$$ {}^{A}_{B}R = \begin{bmatrix} \hat{X}_B \cdot \hat{X}_A & \hat{Y}_B \cdot \hat{X}_A & \hat{Z}_B \cdot \hat{X}_A \\ \hat{X}_B \cdot \hat{Y}_A & \hat{Y}_B \cdot \hat{Y}_A & \hat{Z}_B \cdot \hat{Y}_A \\ \hat{X}_B \cdot \hat{Z}_A & \hat{Y}_B \cdot \hat{Z}_A & \hat{Z}_B \cdot \hat{Z}_A \end{bmatrix} = \begin{bmatrix} | & | & | \\ {}^{A}\hat{X}_B & {}^{A}\hat{Y}_B & {}^{A}\hat{Z}_B \\ | & | & | \end{bmatrix} $$

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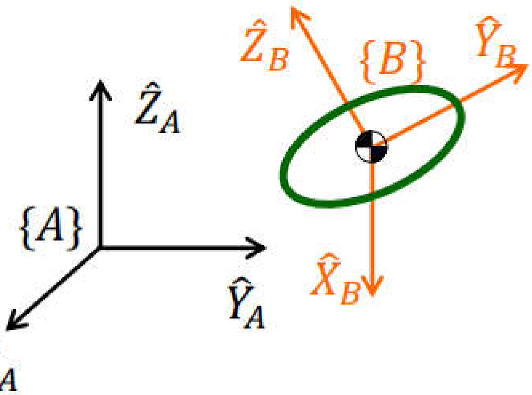

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- 彙整後：

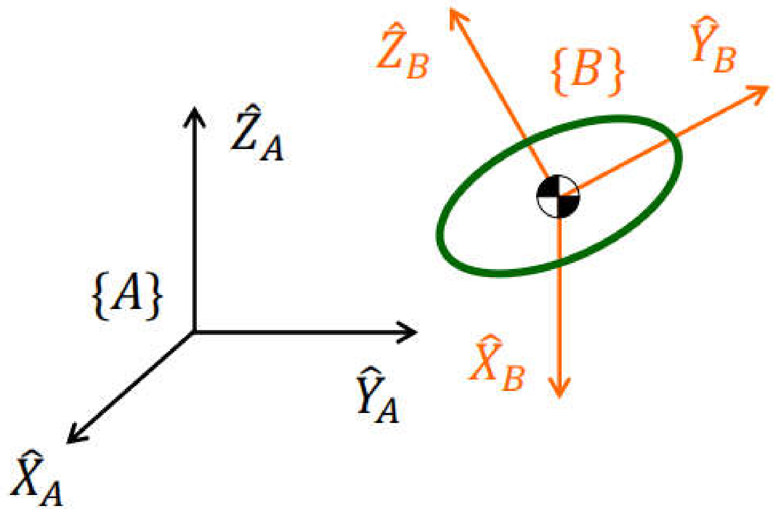
$$\{B\} = \{^{A}_{B}R,\ ^{A}P_{B\ org}\}$$

但無法進行量化計算

2.6 齐次变换矩阵

- 如何將移動和轉動整合在一起描述？

⇨ Homogeneous transformation matrix



2.6 齐次变换矩阵

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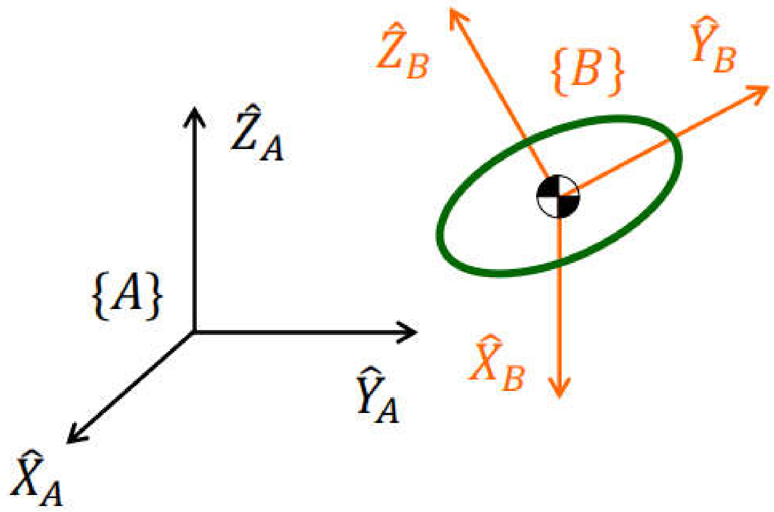
  ⇨ Homogeneous transformation matrix

$$\left[\begin{array}{ccc|c} & {}^{A}_{B}R_{3\times 3} & & {}^{A}P_{B\,org\,3\times 1} \\ \hline 0 & 0 & 0 & 1 \end{array}\right]_{4\times 4}$$



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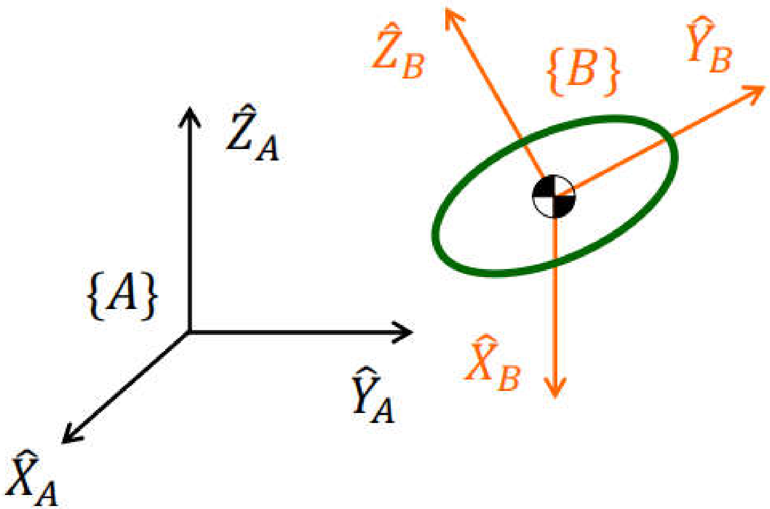
$$\left[\begin{array}{ccc|c} & {}^A_B R_{3\times 3} & & {}^A P_{B\,org\;3\times 1} \\ \hline 0 & 0 & 0 & 1 \end{array}\right]_{4\times 4}$$

$$= \left[\begin{array}{ccc|c} | & | & | & | \\ {}^A\hat{X}_B & {}^A\hat{Y}_B & {}^A\hat{Z}_B & {}^A P_{B\,org} \\ | & | & | & | \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$



2.6 齐次变换矩阵

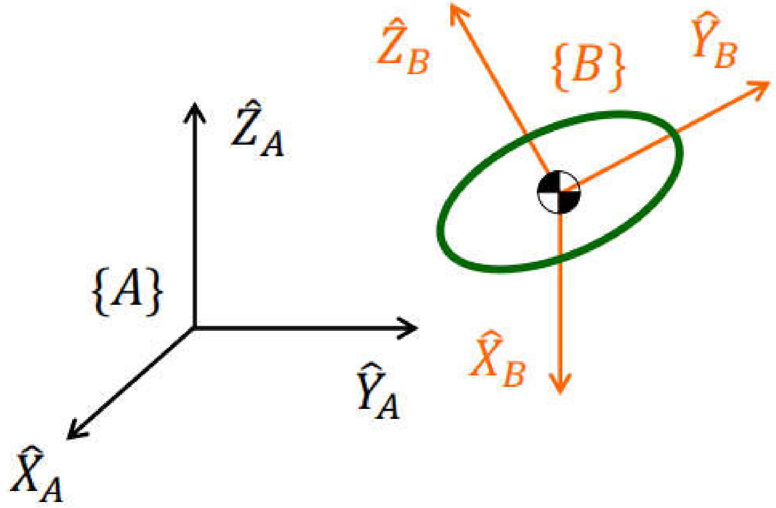
- 如何將移動和轉動整合在一起描述？

⇨ Homogeneous transformation matrix

$$\begin{bmatrix} {}^{A}_{B}R_{3\times 3} & {}^{A}P_{B\,org\,3\times 1} \\ 0\quad 0\quad 0 & 1 \end{bmatrix}_{4\times 4}$$

$$= \begin{bmatrix} | & | & | & | \\ {}^{A}\hat{X}_B & {}^{A}\hat{Y}_B & {}^{A}\hat{Z}_B & {}^{A}P_{B\,org} \\ | & | & | & | \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= {}^{A}_{B}T$$



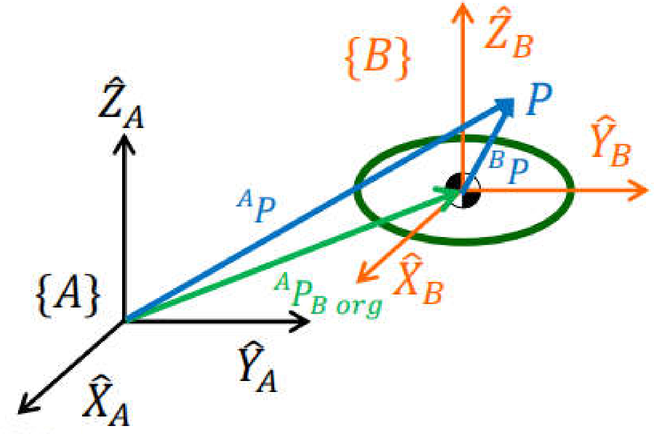
2.6 齐次变换矩阵

- 以Mapping，轉換向量（或點）之座標系的方式來確認${}^{A}_{B}T$運算之正確性

2.6 齐次变换矩阵

- 以Mapping，轉換向量（或點）之座標系的方式來確認$^{A}_{B}T$運算之正確性
  - 僅有 移動

$$^{A}P_{3\times1} = {}^{B}P_{3\times1} + {}^{A}P_{B\,org\;3\times1}$$



2.6 齐次变换矩阵

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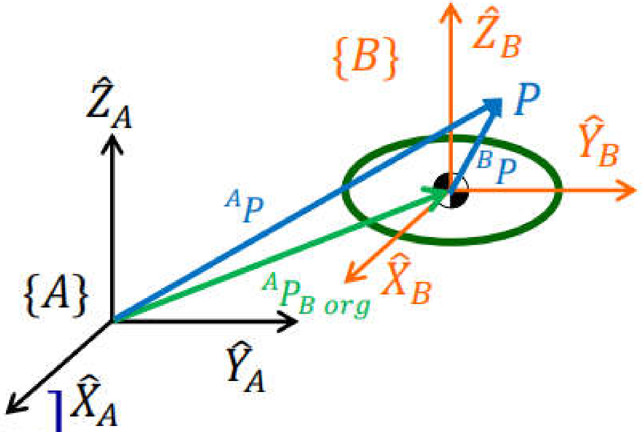
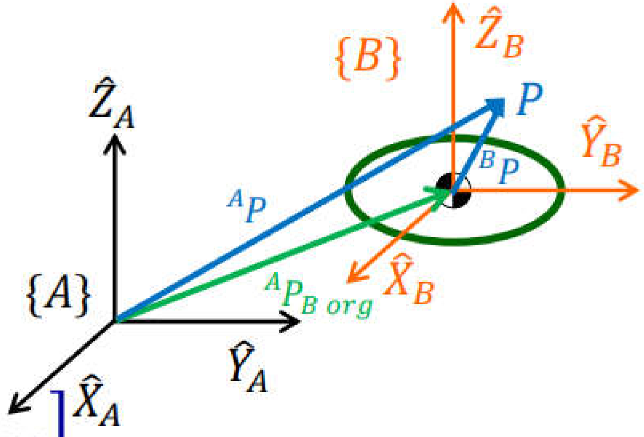


$${}^AP_{3\times1} = {}^BP_{3\times1} + {}^AP_{B\,org\;3\times1}$$

$$\begin{bmatrix} {}^AP \\ 1 \end{bmatrix} = \left[\begin{array}{ccc|c} & I_{3\times3} & & {}^AP_{B\,org} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \begin{bmatrix} {}^BP \\ 1 \end{bmatrix} = \begin{bmatrix} {}^BP + {}^AP_{B\,org} \\ 1 \end{bmatrix}$$

2.6 齐次变换矩阵

- 以Mapping，轉換向量（或點）之座標系的方式來確認${}^A_BT$運算之正確性



  - 僅有 移動

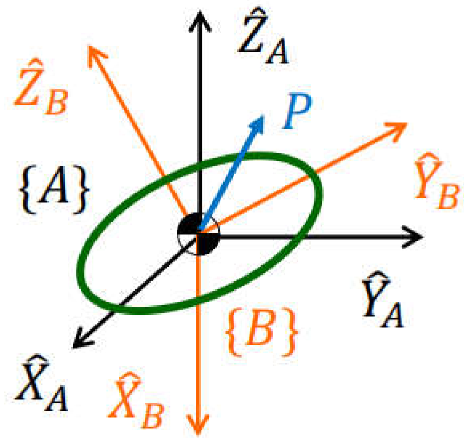
$$ {}^AP_{3\times1} = {}^BP_{3\times1} + {}^AP_{B\,org\;3\times1} $$

$$ \begin{bmatrix} {}^AP \\ 1 \end{bmatrix} = \left[\begin{array}{ccc|c} & I_{3\times3} & & {}^AP_{B\,org} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \begin{bmatrix} {}^BP \\ 1 \end{bmatrix} = \begin{bmatrix} {}^BP + {}^AP_{B\,org} \\ 1 \end{bmatrix} $$

  - 僅有 轉動

$$ {}^AP_{3\times1} = \underset{3\times3}{{}^A_BR}\;{}^BP_{3\times1} $$

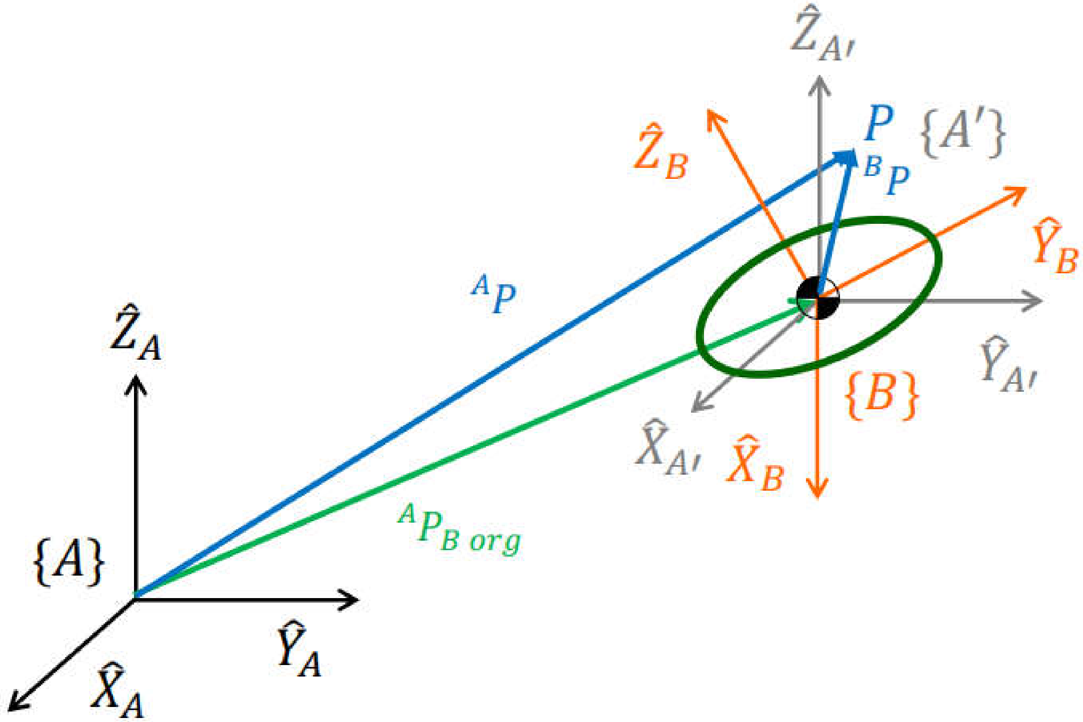
$$ \begin{bmatrix} {}^AP \\ 1 \end{bmatrix} = \left[\begin{array}{ccc|c} & & & 0 \\ & {}^A_BR & & 0 \\ & & & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \begin{bmatrix} {}^BP \\ 1 \end{bmatrix} = \begin{bmatrix} {}^A_BR\,{}^BP \\ 1 \end{bmatrix} $$



2.6 齐次变换矩阵

- 移動和轉動複合

$$ {}^{A}P_{3\times 1} = {}^{A}_{B}R\, {}^{B}P_{3\times 1} + {}^{A}P_{B\,org\;3\times 1} $$

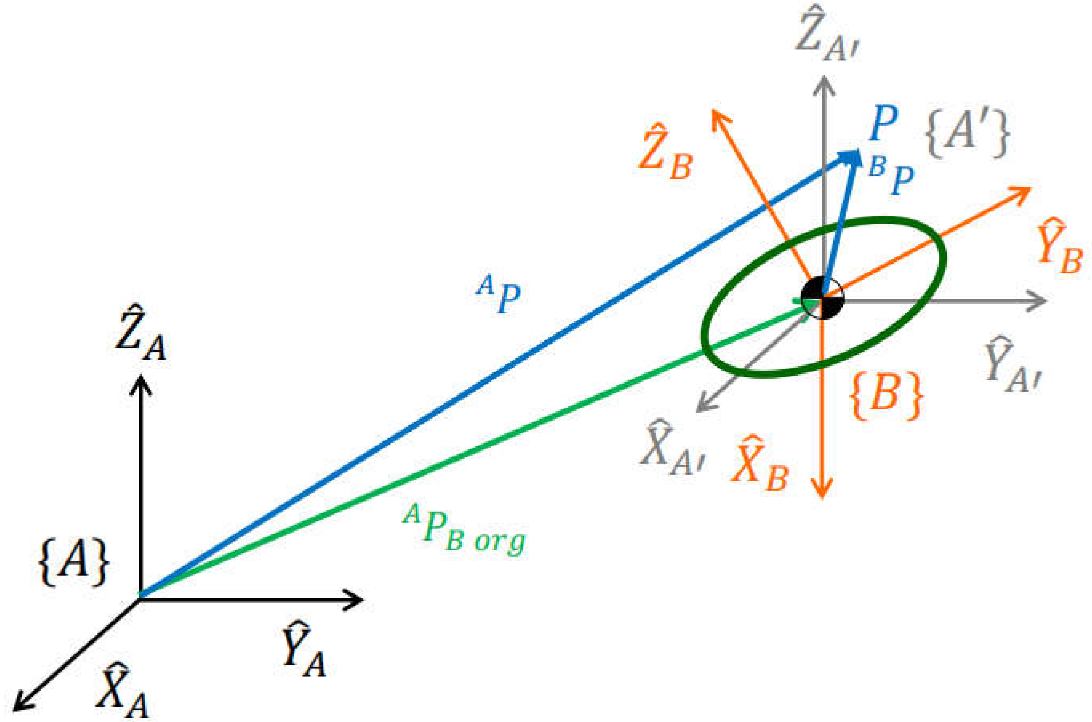


2.6 齐次变换矩阵

- 移動和轉動複合

$$^{A}P_{3\times 1} = {}^{A}_{B}R\, {}^{B}P_{3\times 1} + {}^{A}P_{B\,org\ 3\times 1}$$

$$\begin{bmatrix} ^{A}P \\ 1 \end{bmatrix} = \begin{bmatrix} {}^{A}_{B}R & & & ^{A}P_{B\,org} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ^{B}P \\ 1 \end{bmatrix} = \begin{bmatrix} {}^{A}_{B}R\, {}^{B}P + {}^{A}P_{B\,org} \\ 1 \end{bmatrix}$$



2.6 齐次变换矩阵

- 移動和轉動複合

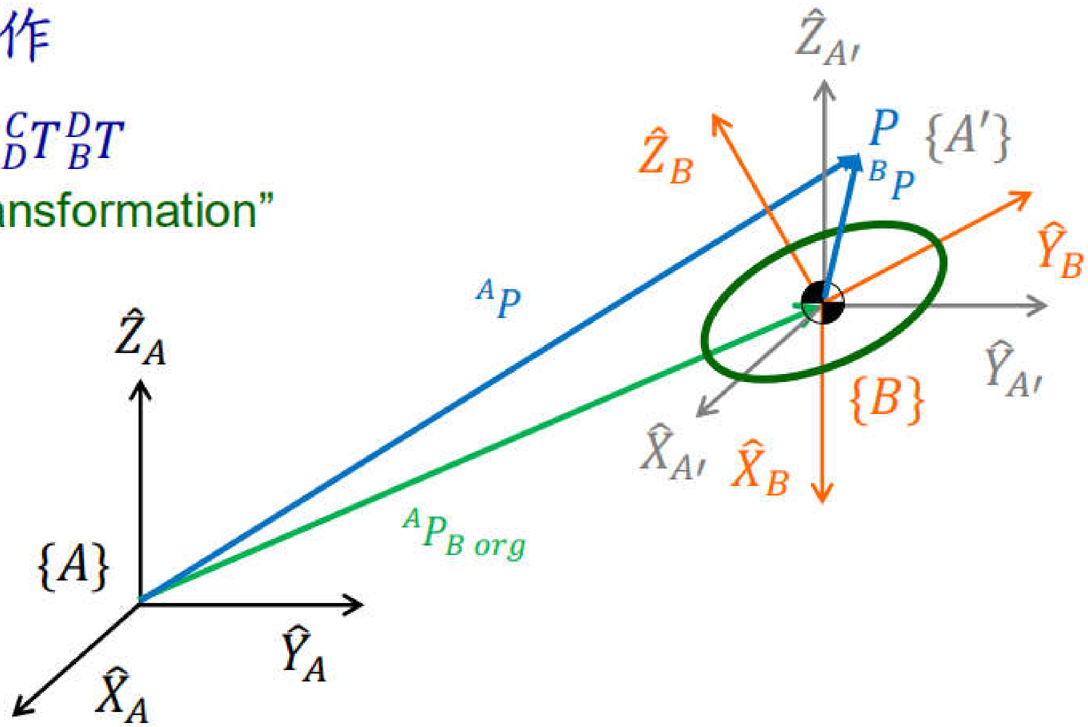
$$ {}^{A}P_{3\times 1} = {}^{A}_{B}R\, {}^{B}P_{3\times 1} + {}^{A}P_{B\,org\ 3\times 1} $$

$$ \begin{bmatrix} {}^{A}P \\ 1 \end{bmatrix} = \begin{bmatrix} {}^{A}_{B}R & {}^{A}P_{B\,org} \\ 0\quad 0\quad 0 & 1 \end{bmatrix} \begin{bmatrix} {}^{B}P \\ 1 \end{bmatrix} = \begin{bmatrix} {}^{A}_{B}R\, {}^{B}P + {}^{A}P_{B\,org} \\ 1 \end{bmatrix} $$

- 可連續操作

$$ {}^{A}_{B}T = {}^{A}_{C}T\, {}^{C}_{D}T\, {}^{D}_{B}T $$

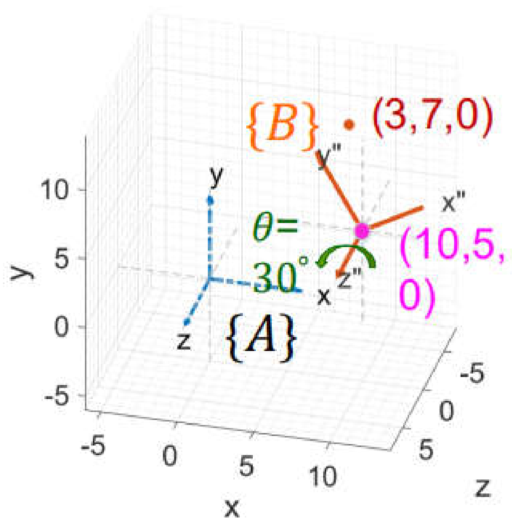
"sequential transformation"



2.6 齐次变换矩阵

- Ex:

$$^{B}P=\begin{bmatrix}3\\7\\0\end{bmatrix}\quad ^{A}P_{Borg}=\begin{bmatrix}10\\5\\0\end{bmatrix}\quad ^{A}\hat{X}_{B}=\begin{bmatrix}\frac{\sqrt{3}}{2}\\\frac{1}{2}\\0\end{bmatrix}\quad ^{A}\hat{Y}_{B}=\begin{bmatrix}-\frac{1}{2}\\\frac{\sqrt{3}}{2}\\0\end{bmatrix}\quad ^{A}\hat{Z}_{B}=\begin{bmatrix}0\\0\\1\end{bmatrix}\Rightarrow\ ^{A}P=?$$

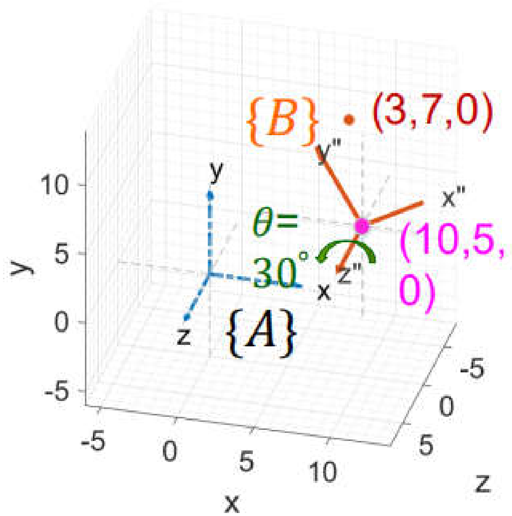


2.6 齐次变换矩阵

- Ex:

$${}^{B}P = \begin{bmatrix} 3 \\ 7 \\ 0 \end{bmatrix} \quad {}^{A}P_{Borg} = \begin{bmatrix} 10 \\ 5 \\ 0 \end{bmatrix} \quad {}^{A}\hat{X}_{B} = \begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix} \quad {}^{A}\hat{Y}_{B} = \begin{bmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \\ 0 \end{bmatrix} \quad {}^{A}\hat{Z}_{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow {}^{A}P = ?$$

$$\begin{bmatrix} {}^{A}P \\ 1 \end{bmatrix} = \begin{bmatrix} {}^{A}_{B}R & {}^{A}P_{B\,org} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix} \begin{bmatrix} {}^{B}P \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 & 10 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 7 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 9.098 \\ 12.562 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} | \\ {}^{A}P \\ | \\ 1 \end{bmatrix}$$



2.6 齐次变换矩阵

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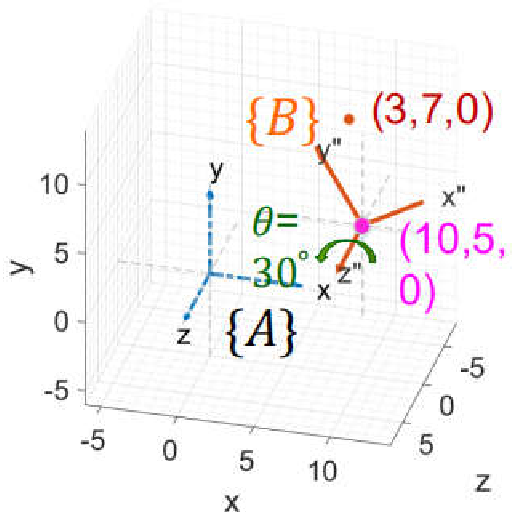
$$^{B}P=\begin{bmatrix}3\\7\\0\end{bmatrix}\quad ^{A}P_{Borg}=\begin{bmatrix}10\\5\\0\end{bmatrix}\quad ^{A}\hat{X}_B=\begin{bmatrix}\frac{\sqrt{3}}{2}\\\frac{1}{2}\\0\end{bmatrix}\quad ^{A}\hat{Y}_B=\begin{bmatrix}-\frac{1}{2}\\\frac{\sqrt{3}}{2}\\0\end{bmatrix}\quad ^{A}\hat{Z}_B=\begin{bmatrix}0\\0\\1\end{bmatrix}\Rightarrow\ ^{A}P=?$$

$$\begin{bmatrix}^{A}P\\1\end{bmatrix}=\begin{bmatrix}^{A}_{B}R & ^{A}P_{B\,org}\\0\ \ 0\ \ 0 & 1\end{bmatrix}\begin{bmatrix}^{B}P\\1\end{bmatrix}=\begin{bmatrix}\frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 & 10\\\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 5\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}3\\7\\0\\1\end{bmatrix}=\begin{bmatrix}9.098\\12.562\\0\\1\end{bmatrix}=\begin{bmatrix}|\\^{A}P\\|\\1\end{bmatrix}$$

單純看$^{A}_{B}T$: 表達$\{B\}$相對於$\{A\}$的方法

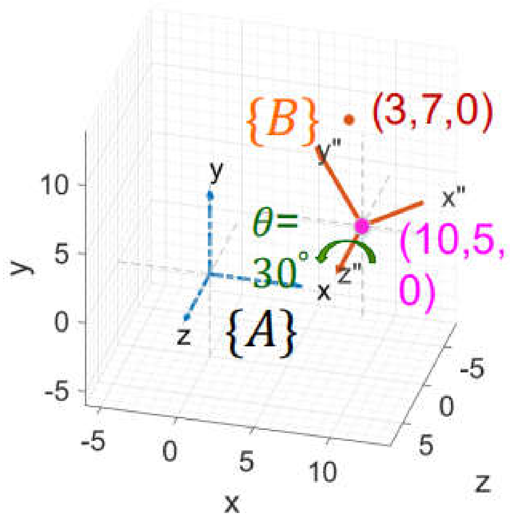


2.6 齐次变换矩阵

- Ex:

$$ {}^{B}P=\begin{bmatrix}3\\7\\0\end{bmatrix}\quad {}^{A}P_{Borg}=\begin{bmatrix}10\\5\\0\end{bmatrix}\quad {}^{A}\hat{X}_{B}=\begin{bmatrix}\frac{\sqrt{3}}{2}\\\frac{1}{2}\\0\end{bmatrix}\quad {}^{A}\hat{Y}_{B}=\begin{bmatrix}-\frac{1}{2}\\\frac{\sqrt{3}}{2}\\0\end{bmatrix}\quad {}^{A}\hat{Z}_{B}=\begin{bmatrix}0\\0\\1\end{bmatrix}\quad \Rightarrow \quad {}^{A}P=? $$

$$ \begin{bmatrix}{}^{A}P\\1\end{bmatrix}=\begin{bmatrix}{}^{A}_{B}R & {}^{A}P_{B\,org}\\0\quad 0\quad 0 & 1\end{bmatrix}\begin{bmatrix}{}^{B}P\\1\end{bmatrix}=\underbrace{\begin{bmatrix}\frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 & 10\\\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 5\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}}\begin{bmatrix}3\\7\\0\\1\end{bmatrix}=\begin{bmatrix}9.098\\12.562\\0\\1\end{bmatrix}=\begin{bmatrix}|\\{}^{A}P\\|\\1\end{bmatrix} $$



單純看${}^{A}_{B}T$: 表達$\{B\}$相對於$\{A\}$的方法

看整個操作：

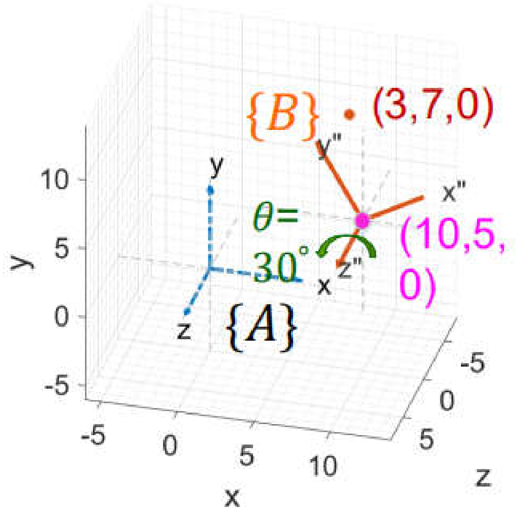
轉換point在不同frame下的表達

2.6 齐次变换矩阵

- Ex:

$$ {}^{B}P = \begin{bmatrix} 3 \\ 7 \\ 0 \end{bmatrix} \quad {}^{A}P_{Borg} = \begin{bmatrix} 10 \\ 5 \\ 0 \end{bmatrix} \quad {}^{A}\hat{X}_{B} = \begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix} \quad {}^{A}\hat{Y}_{B} = \begin{bmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \\ 0 \end{bmatrix} \quad {}^{A}\hat{Z}_{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \quad {}^{A}P = ? $$

$$ \begin{bmatrix} {}^{A}P \\ 1 \end{bmatrix} = \begin{bmatrix} {}^{A}_{B}R & {}^{A}P_{B\,org} \\ 0\quad 0\quad 0 & 1 \end{bmatrix} \begin{bmatrix} {}^{B}P \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 & 10 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 7 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 9.098 \\ 12.562 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} | \\ {}^{A}P \\ | \\ 1 \end{bmatrix} $$

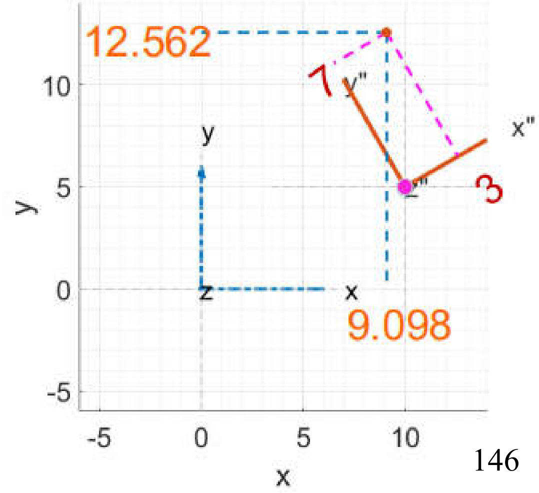


單純看 ${}^{A}_{B}T$: 表達{B}相對於{A}的方法

看整個操作：

轉換point在不同frame下的表達

投影至XY平面驗證答案



2.6 齐次变换矩阵

- $^{A}_{B}T$除了Mapping之外，也可當Operator，對向量（或點）進行移動或轉動

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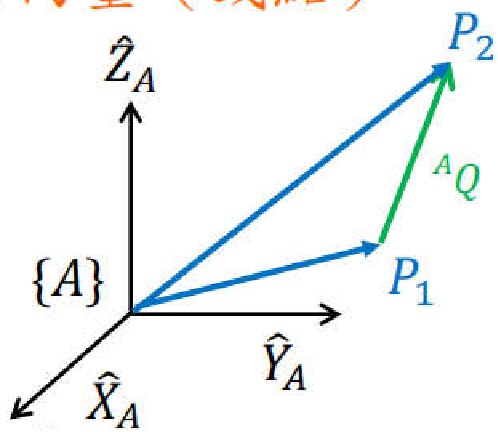
- ${}^{A}_{B}T$除了Mapping之外，也可當Operator，對向量（或點）進行移動或轉動
  - 僅有 移動

$${}^{A}P_{2_{3\times1}} = {}^{A}P_{1_{3\times1}} + {}^{A}Q_{3\times1}$$



2.6 齐次变换矩阵

- ${}^{A}_{B}T$除了Mapping之外，也可當Operator，對向量（或點）進行移動或轉動
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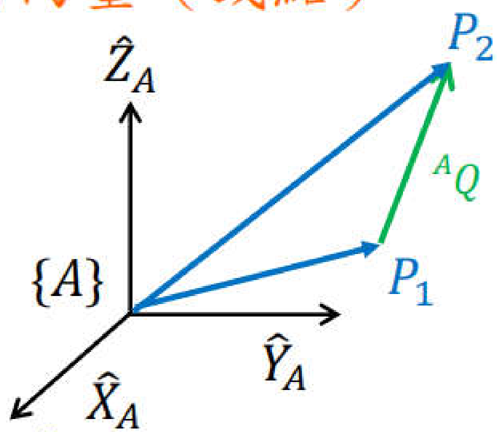
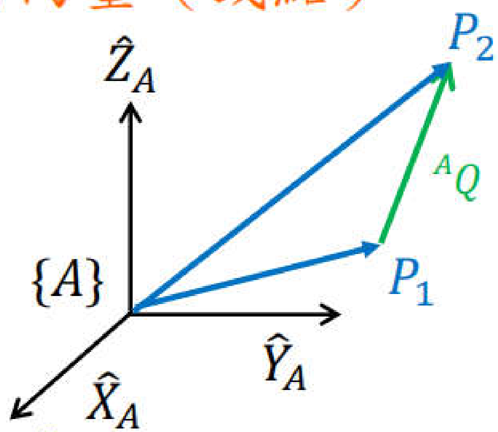


$${}^{A}P_{2_{3\times 1}} = {}^{A}P_{1_{3\times 1}} + {}^{A}Q_{3\times 1}$$

$$\begin{bmatrix} {}^{A}P_2 \\ 1 \end{bmatrix} = D(Q)\begin{bmatrix} {}^{A}P_1 \\ 1 \end{bmatrix} = \begin{bmatrix} I & & {}^{A}Q \\ 0 \quad 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} {}^{A}P_1 \\ 1 \end{bmatrix} = \begin{bmatrix} {}^{A}P_1 + {}^{A}Q \\ 1 \end{bmatrix}$$

2.6 齐次变换矩阵

- ^A_BT除了Mapping之外，也可當Operator，對向量（或點）進行移動或轉動



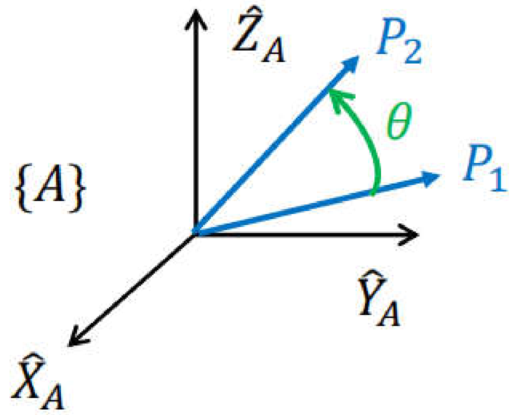
  - 僅有 移動

$$^AP_{2_{3\times1}} = {^AP_1}_{3\times1} + {^AQ}_{3\times1}$$

$$\begin{bmatrix} ^AP_2 \\ 1 \end{bmatrix} = D(Q)\begin{bmatrix} ^AP_1 \\ 1 \end{bmatrix} = \begin{bmatrix} & I & & ^AQ \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} ^AP_1 \\ 1 \end{bmatrix} = \begin{bmatrix} ^AP_1 + {^AQ} \\ 1 \end{bmatrix}$$

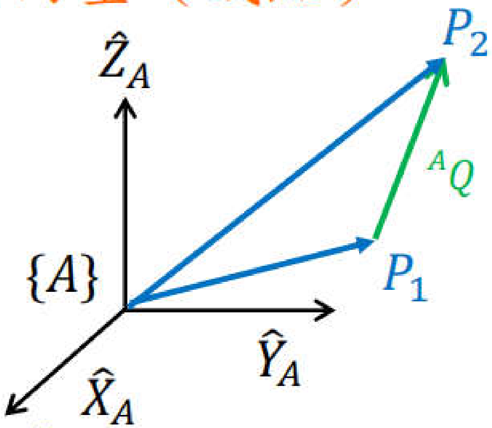
  - 僅有 轉動

$$^AP_{2_{3\times1}} = \underset{3\times3}{R_{\hat{K}}(\theta)}\, {^AP_1}_{3\times1}$$



2.6 齐次变换矩阵

- ${}^{A}_{B}T$除了Mapping之外，也可當Operator，對向量（或點）進行移動或轉動



  - 僅有 移動

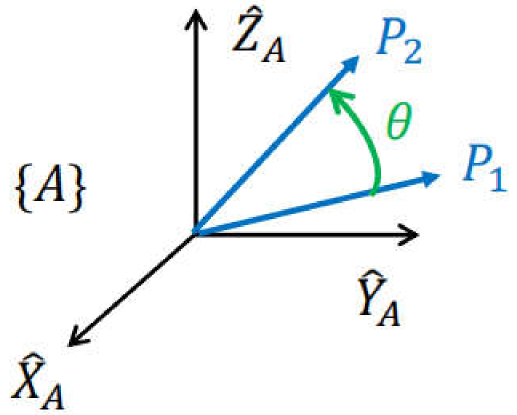
$$ {}^{A}P_{2\,3\times 1} = {}^{A}P_{1\,3\times 1} + {}^{A}Q_{3\times 1} $$

$$ \begin{bmatrix} {}^{A}P_2 \\ 1 \end{bmatrix} = D(Q) \begin{bmatrix} {}^{A}P_1 \\ 1 \end{bmatrix} = \begin{bmatrix} & I & & {}^{A}Q \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} {}^{A}P_1 \\ 1 \end{bmatrix} = \begin{bmatrix} {}^{A}P_1 + {}^{A}Q \\ 1 \end{bmatrix} $$

  - 僅有 轉動

$$ {}^{A}P_{2\,3\times 1} = \underset{3\times 3}{R_{\hat{K}}(\theta)}\, {}^{A}P_{1\,3\times 1} $$

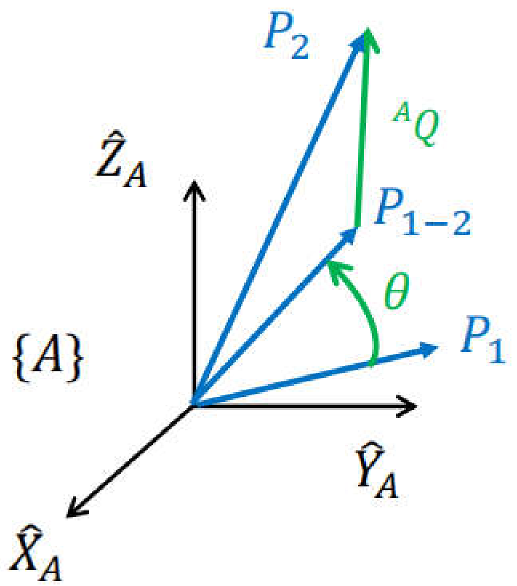
$$ \begin{bmatrix} {}^{A}P_2 \\ 1 \end{bmatrix} = \begin{bmatrix} & & & 0 \\ & R_{\hat{K}}(\theta) & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} {}^{A}P_1 \\ 1 \end{bmatrix} = \begin{bmatrix} R_{\hat{K}}(\theta)\, {}^{A}P_1 \\ 1 \end{bmatrix} $$



2.6 齐次变换矩阵

◆ 移動和轉動複合

$$ {}^{A}P_{2_{3\times1}} = \underset{3\times3}{R_{\hat{K}}(\theta)}\,{}^{A}P_{1_{3\times1}} + {}^{A}Q_{3\times1} $$ 先轉動再移動



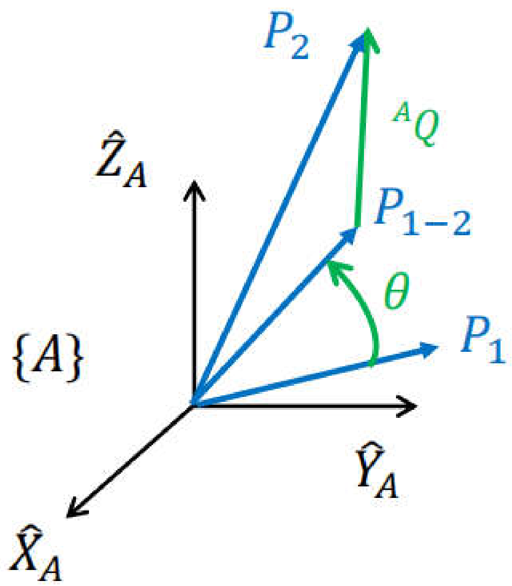
先轉動再移動

2.6 齐次变换矩阵

- 移動和轉動複合

$$ {}^{A}P_{2\,3\times1} = R_{\hat{K}\,3\times3}(\theta)\, {}^{A}P_{1\,3\times1} + {}^{A}Q_{3\times1} $$ 先轉動再移動

$$ \begin{bmatrix} {}^{A}P_2 \\ 1 \end{bmatrix} = \begin{bmatrix} R_{\hat{K}}(\theta) & & & {}^{A}Q \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} {}^{A}P_1 \\ 1 \end{bmatrix} = \begin{bmatrix} R_{\hat{K}}(\theta)\, {}^{A}P_1 + {}^{A}Q \\ 1 \end{bmatrix} = T \begin{bmatrix} {}^{A}P_1 \\ 1 \end{bmatrix} $$



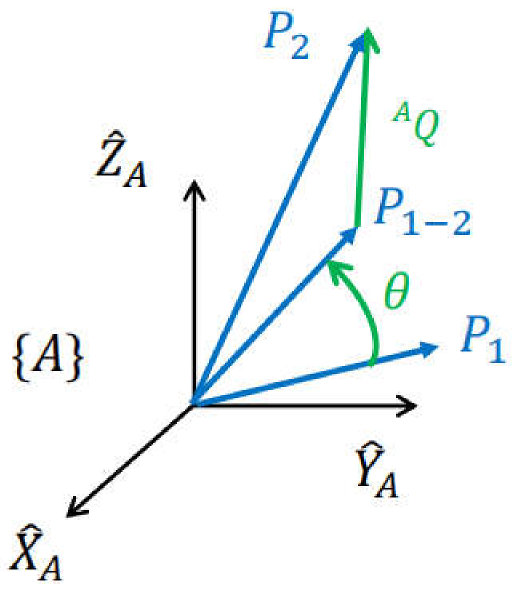
先轉動再移動

2.6 齐次变换矩阵

- 移動和轉動複合

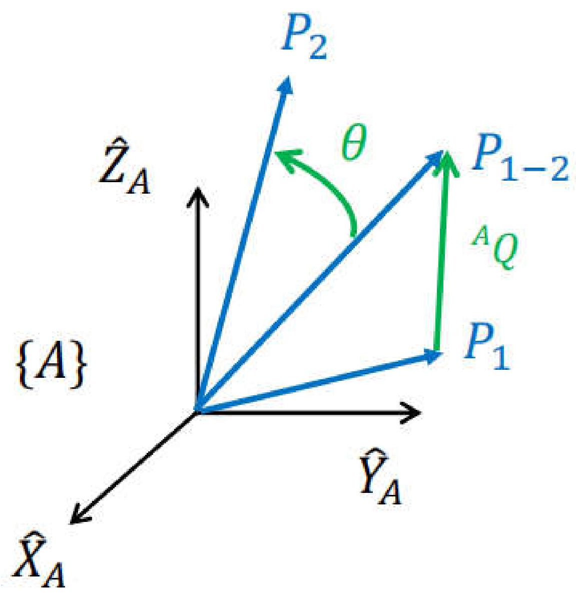
$$ {}^{A}P_{2_{3\times 1}} = \underset{3\times 3}{R_{\hat{K}}(\theta)}\,{}^{A}P_{1_{3\times 1}} + {}^{A}Q_{3\times 1} $$ 先轉動再移動

$$ \begin{bmatrix} {}^{A}P_2 \\ 1 \end{bmatrix} = \begin{bmatrix} R_{\hat{K}}(\theta) & & & {}^{A}Q \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} {}^{A}P_1 \\ 1 \end{bmatrix} = \begin{bmatrix} R_{\hat{K}}(\theta)\,{}^{A}P_1 + {}^{A}Q \\ 1 \end{bmatrix} = T \begin{bmatrix} {}^{A}P_1 \\ 1 \end{bmatrix} $$



先轉動再移動

$\neq$

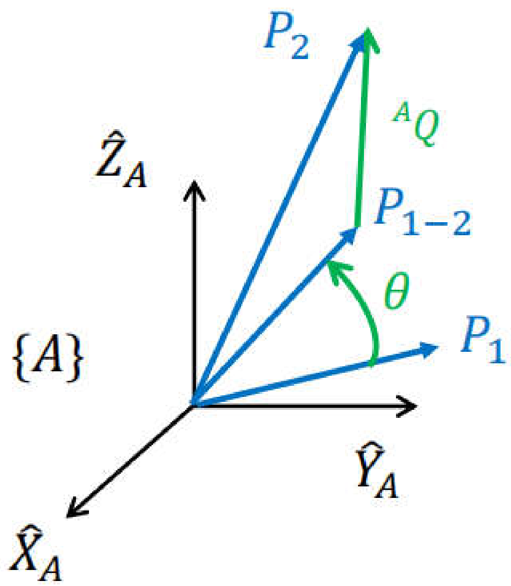


2.6 齐次变换矩阵

- 移動和轉動複合

$${}^{A}P_{2\,3\times1} = R_{\hat{K}}(\theta)_{3\times3}\,{}^{A}P_{1\,3\times1} + {}^{A}Q_{3\times1}$$ 先轉動再移動

$$\begin{bmatrix} {}^{A}P_2 \\ 1 \end{bmatrix} = \begin{bmatrix} R_{\hat{K}}(\theta) & & & {}^{A}Q \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} {}^{A}P_1 \\ 1 \end{bmatrix} = \begin{bmatrix} R_{\hat{K}}(\theta)\,{}^{A}P_1 + {}^{A}Q \\ 1 \end{bmatrix} = T \begin{bmatrix} {}^{A}P_1 \\ 1 \end{bmatrix}$$



先轉動再移動

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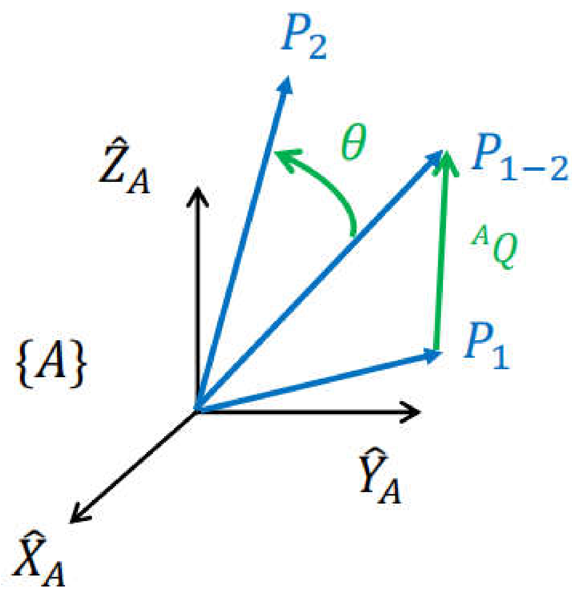

先移動再轉動（${}^{A}Q$ 也會被轉動到）

$${}^{A}P_2 = R_{\hat{K}}(\theta)\left({}^{A}P_1 + {}^{A}Q\right) = R_{\hat{K}}(\theta)\,{}^{A}P_1 + R_{\hat{K}}(\theta)\,{}^{A}Q$$

2.6 齐次变换矩阵

- Ex: Point $P_1 = \begin{bmatrix} 3 \\ 7 \\ 0 \end{bmatrix}$，先對Z軸CCW轉$30^\circ$, 然後移動$\begin{bmatrix} 10 \\ 5 \\ 0 \end{bmatrix}$到$P_2$ ⇨ $P_2 = ?$

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和「Mapping -3」的答案相同，Why?

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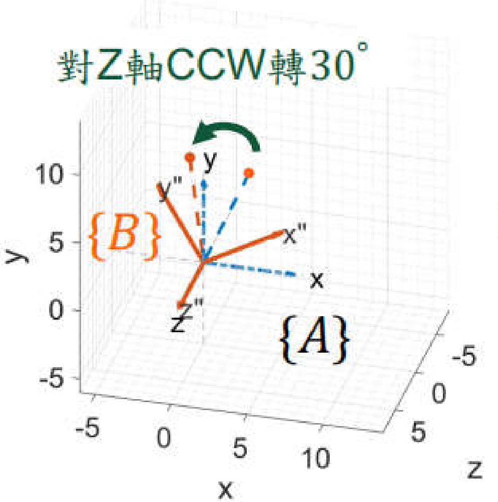

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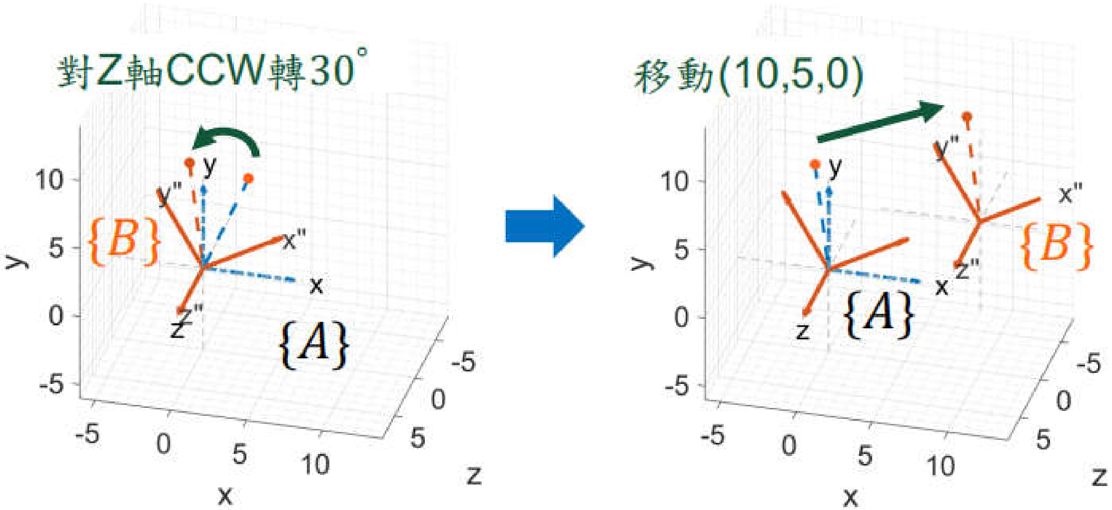

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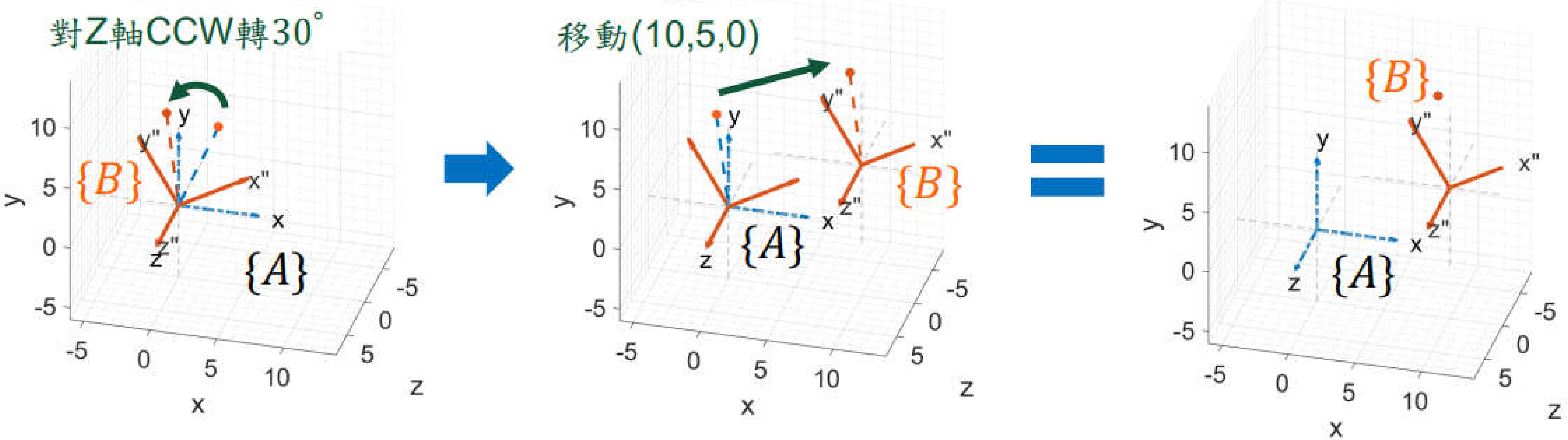
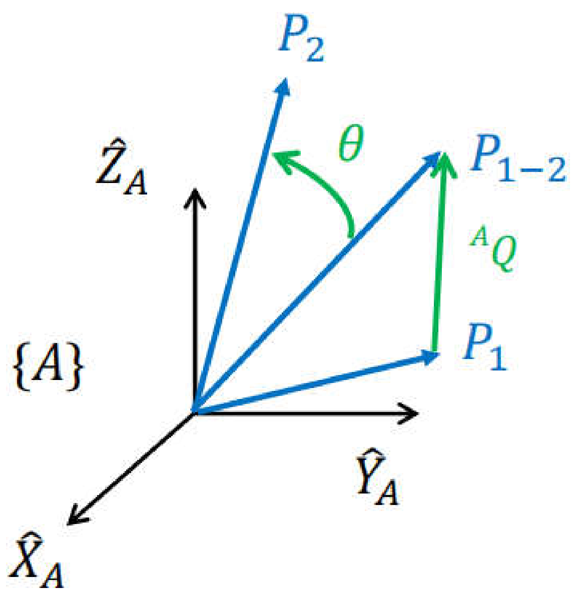


2.6 齐次变换矩阵

- In-video Quiz: 如果要如下圖所示的先移動再轉動，那T應該如何表達？



A. $\begin{bmatrix} R_{\hat{K}}(\theta) & {}^AQ \\ 0\quad 0\quad 0 & 1 \end{bmatrix}$

B. $\begin{bmatrix} I & {}^AQ \\ 0\quad 0\quad 0 & 1 \end{bmatrix}\begin{bmatrix} R_{\hat{K}}(\theta) & \begin{matrix}0\\0\\0\end{matrix} \\ 0\quad 0\quad 0 & 1 \end{bmatrix}$

C. $\begin{bmatrix} R_{\hat{K}}(\theta) & \begin{matrix}0\\0\\0\end{matrix} \\ 0\quad 0\quad 0 & 1 \end{bmatrix}\begin{bmatrix} I & {}^AQ \\ 0\quad 0\quad 0 & 1 \end{bmatrix}$

D. $\begin{bmatrix} R_{\hat{K}}(\theta) & {}^AQ \\ 0\quad 0\quad 0 & 1 \end{bmatrix}\begin{bmatrix} R_{\hat{K}}(\theta) & \begin{matrix}0\\0\\0\end{matrix} \\ 0\quad 0\quad 0 & 1 \end{bmatrix}$

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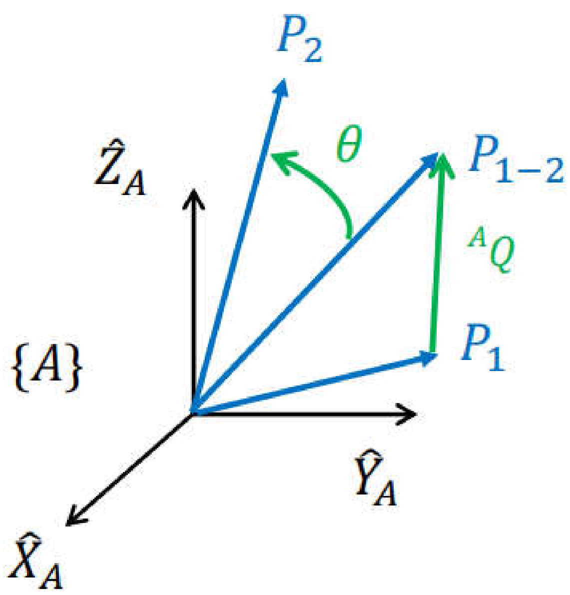

A. $\begin{bmatrix} R_{\hat{K}}(\theta) & & & {}^AQ \\ 0 & 0 & 0 & 1 \end{bmatrix}$

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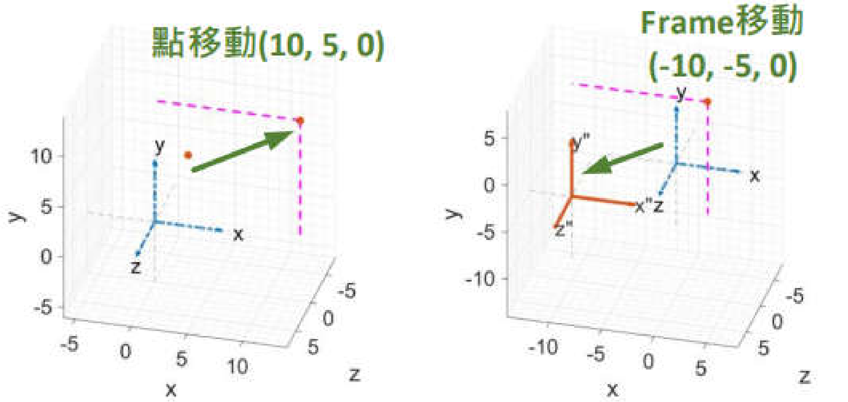
2.6 齐次变换矩阵

- 因為運動是相對的，${}^{A}_{B}T$當Operator時對向量（或點）進行移動或轉動的操作，也可以想成是對frame進行「反向」的移動或轉動的操作
  - Point往前移 = frame往後移



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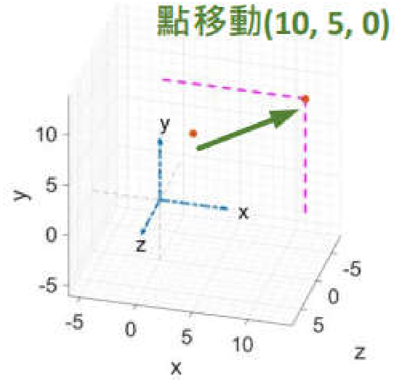
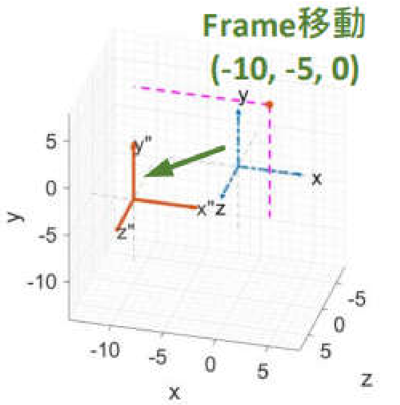
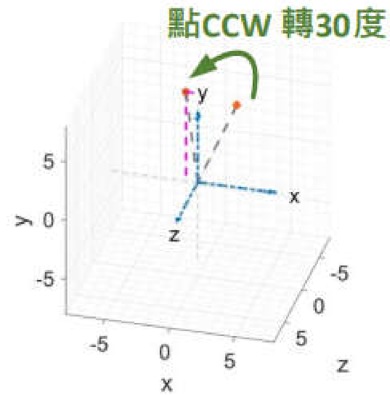
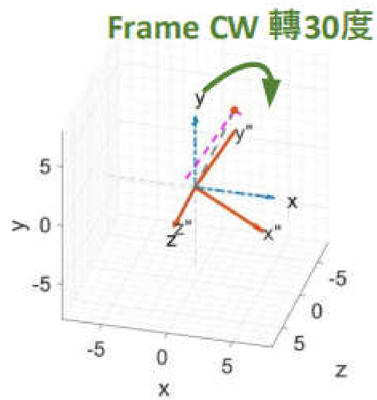




  - Point逆時針轉= frame順時針轉





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- Ex: Revisit Operator-3的範例，改以frame轉動的角度來想

Point $P_1 = \begin{bmatrix} 3 \\ 7 \\ 0 \end{bmatrix}$，先對Z軸CCW轉$30^\circ$, 然後移動$\begin{bmatrix} 10 \\ 5 \\ 0 \end{bmatrix}$到$P_2$ ⇨ $P_2 = ?$

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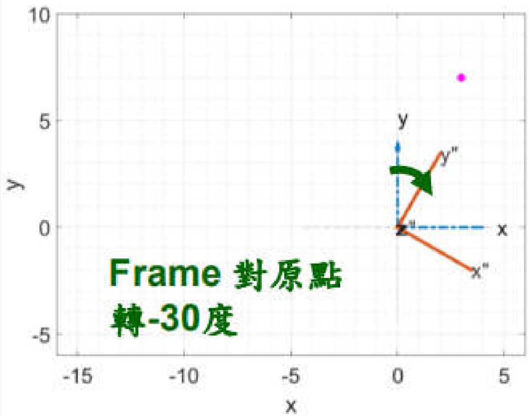
投影至XY平面來看



烟台理工學院 Yantai Institute of Technology

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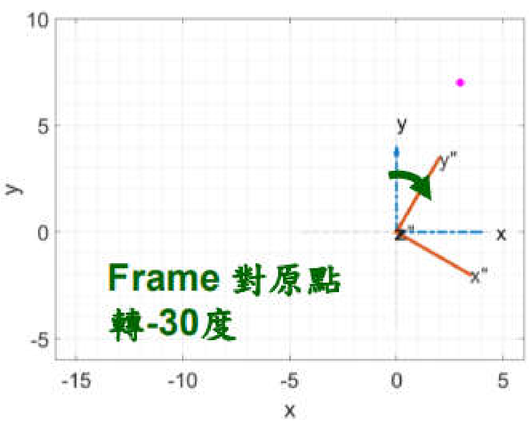

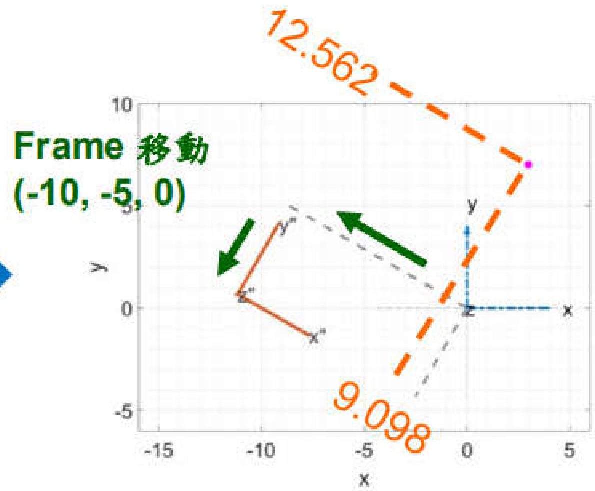

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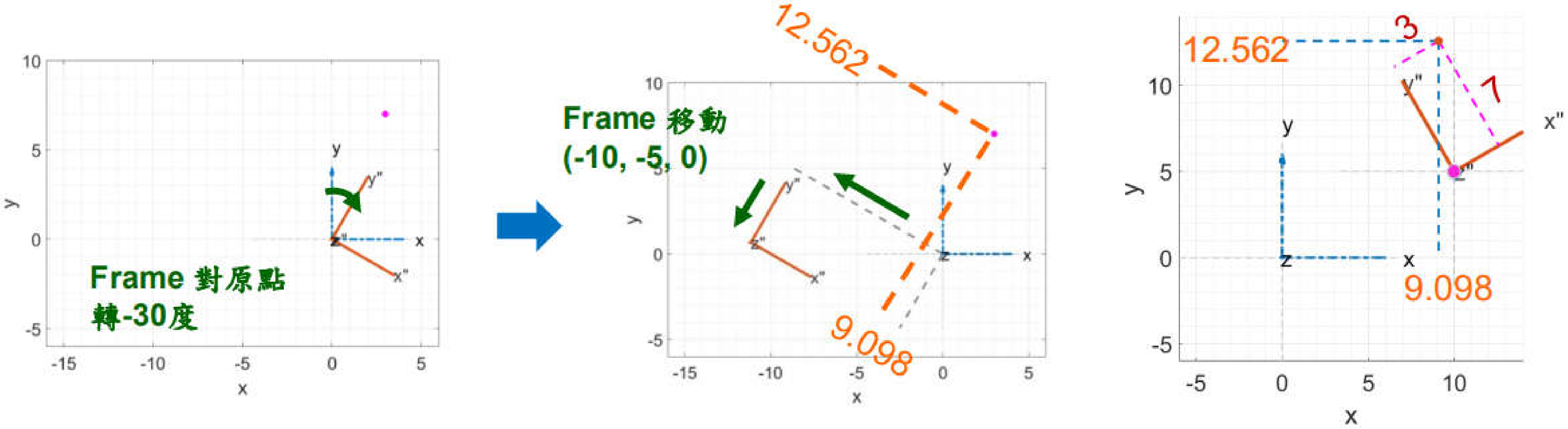
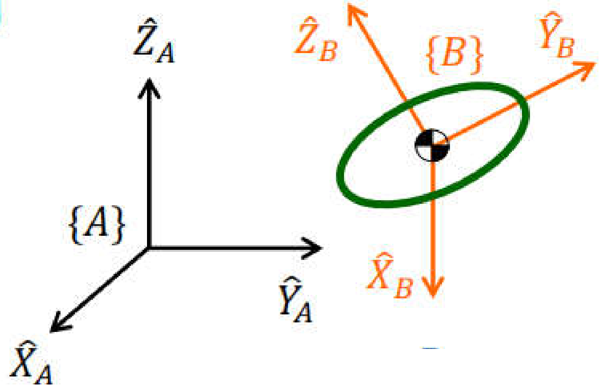


與Operator -3的
答案相同

2.6 齐次变换矩阵

- Homogeneous transformation matrix 的三種用法
  - 描述一個frame(相對於另一個frame)的空間狀態

$$ {}^{A}_{B}T = \begin{bmatrix} | & | & | & | \\ {}^{A}\hat{X}_B & {}^{A}\hat{Y}_B & {}^{A}\hat{Z}_B & {}^{A}P_{B\,org} \\ | & | & | & | \\ 0 & 0 & 0 & 1 \end{bmatrix} $$



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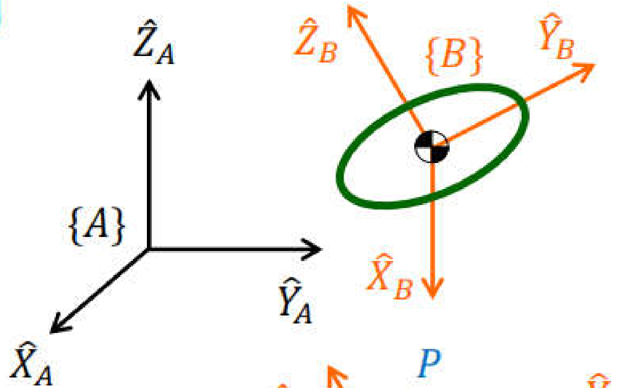
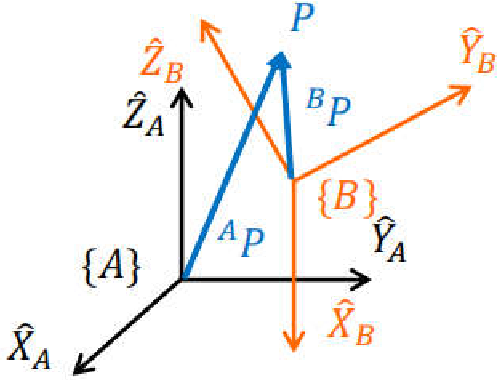


  - 將point由某一個frame的表達換到另一個frame來表達

$$ \begin{bmatrix} {}^{A}P \\ 1 \end{bmatrix} = {}^{A}_{B}T \begin{bmatrix} {}^{B}P \\ 1 \end{bmatrix} $$



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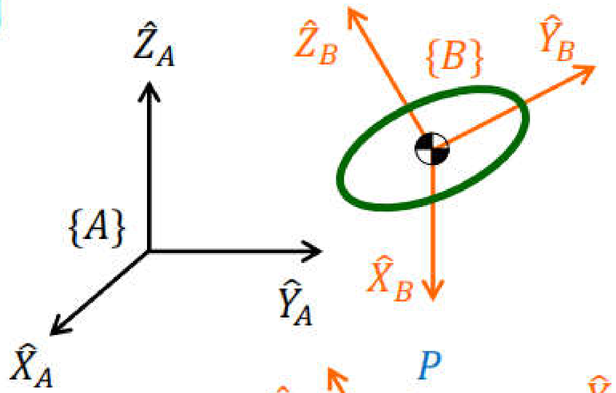

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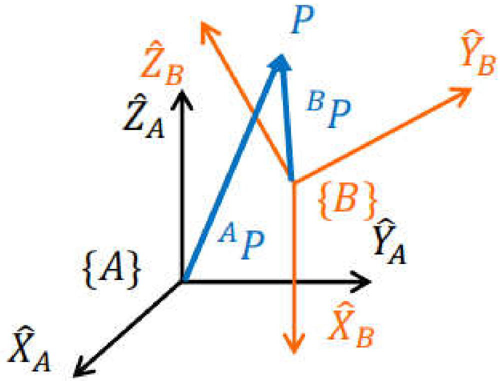
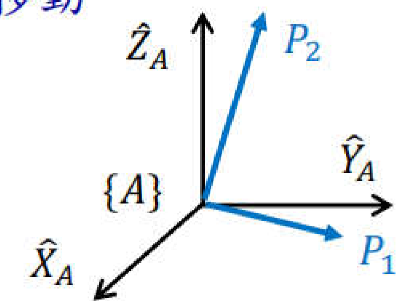


  - 將point(vector)在同一個frame中進行移動和轉動

$$ \begin{bmatrix} {}^{A}P_2 \\ 1 \end{bmatrix} = T \begin{bmatrix} {}^{A}P_1 \\ 1 \end{bmatrix} $$

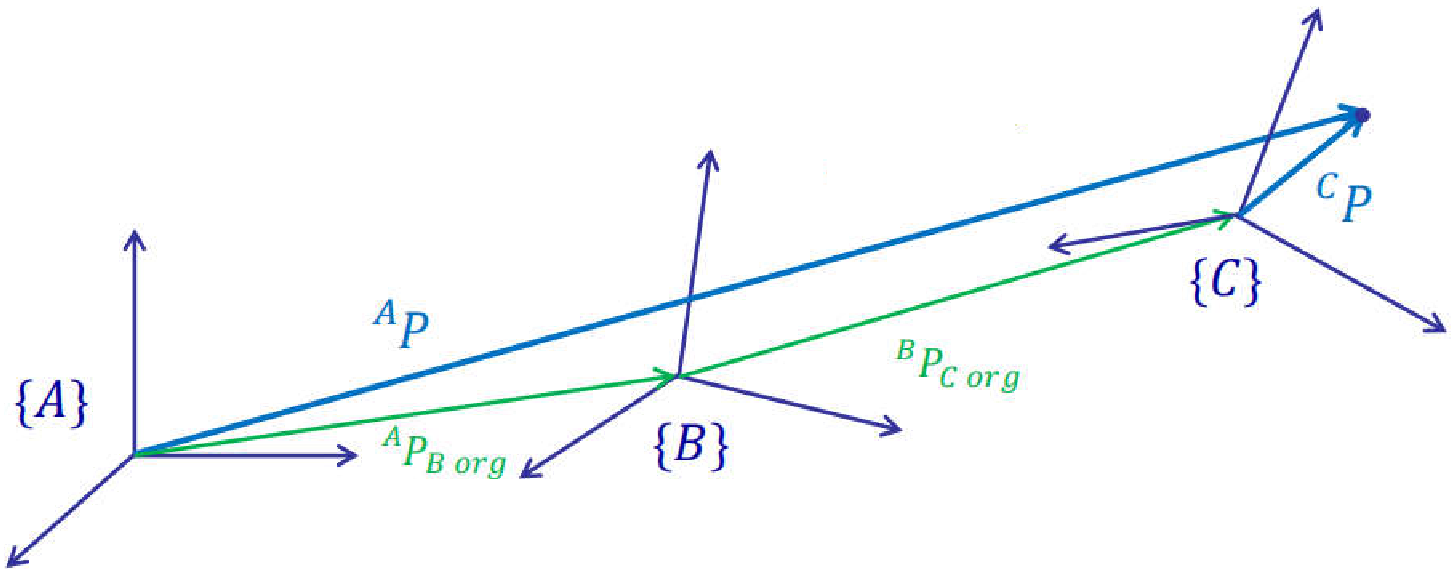


第二章 空间描述和变换

2.7 变换矩阵的运算法则

- 連續運算

$$^{A}P = {}^{A}_{B}T\,{}^{B}P = {}^{A}_{B}T({}^{B}_{C}T\,{}^{C}P) = {}^{A}_{B}T\,{}^{B}_{C}T\,{}^{C}P$$



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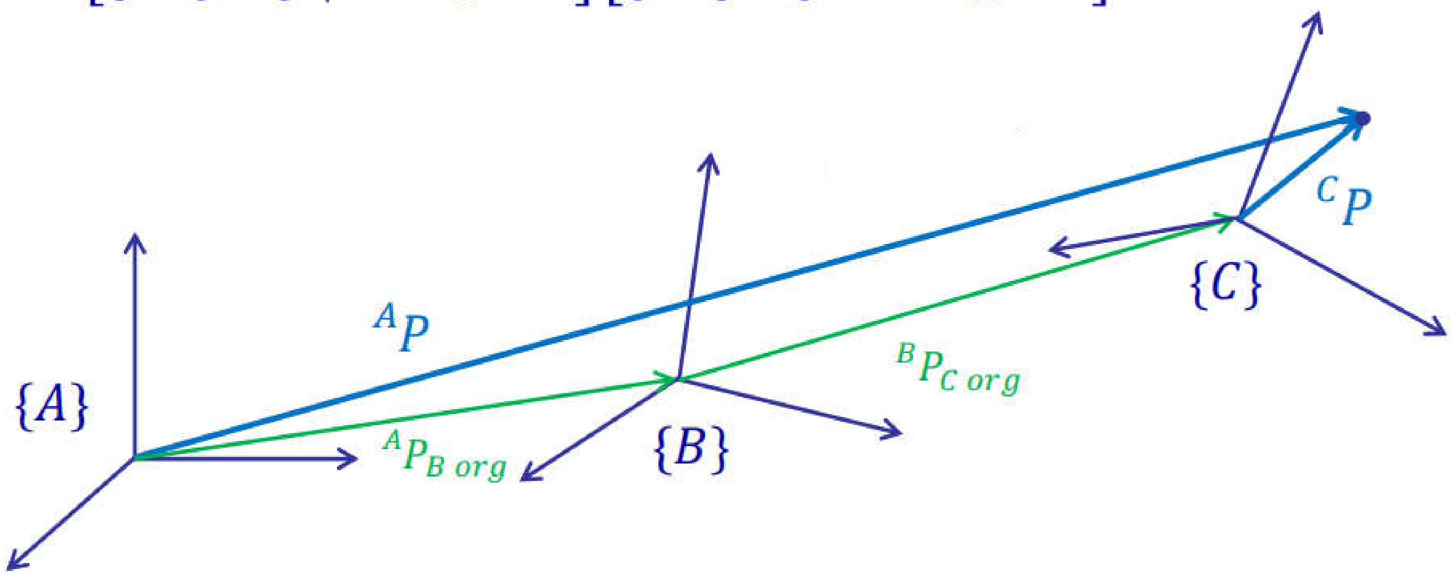
$$ = \left[\begin{array}{ccc|c} & {}^{A}_{B}R & & {}^{A}P_{B\,org} \\ \hline 0 & 0 & 0 & 1 \end{array}\right]\left[\begin{array}{ccc|c} & {}^{B}_{C}R & & {}^{B}P_{C\,org} \\ \hline 0 & 0 & 0 & 1 \end{array}\right]{}^{C}P $$



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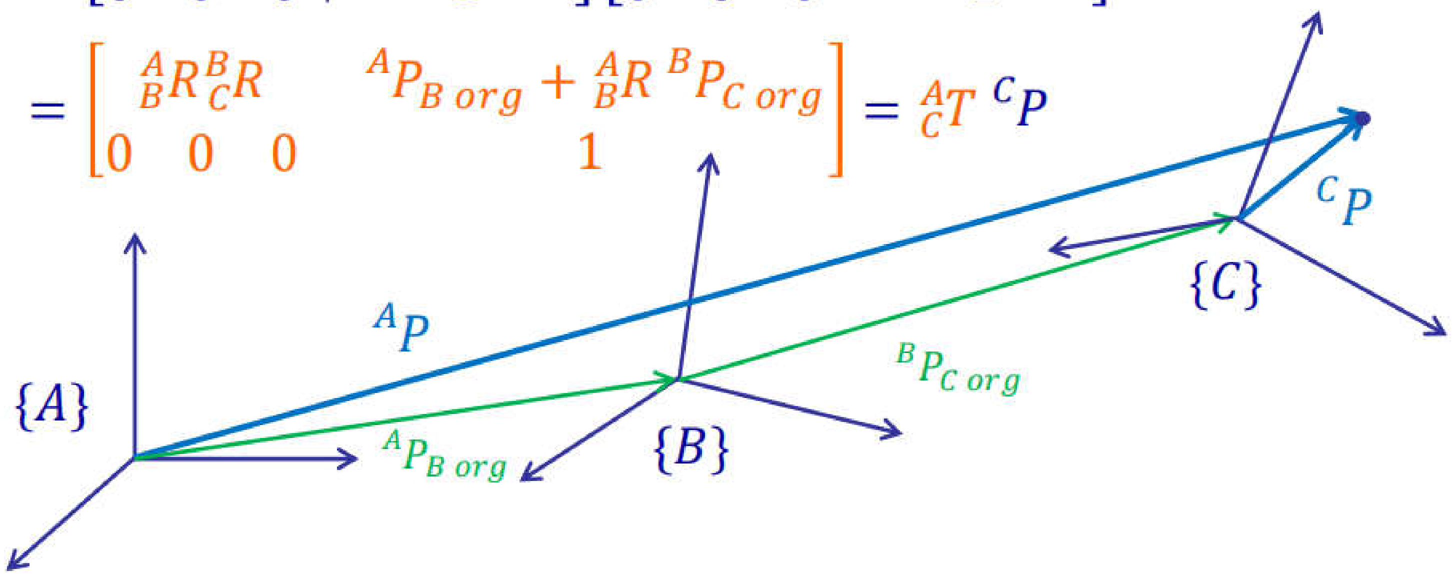
$$ = \begin{bmatrix} {}^{A}_{B}R\,{}^{B}_{C}R & {}^{A}P_{B\,org} + {}^{A}_{B}R\,{}^{B}P_{C\,org} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix} = {}^{A}_{C}T\,{}^{C}P $$



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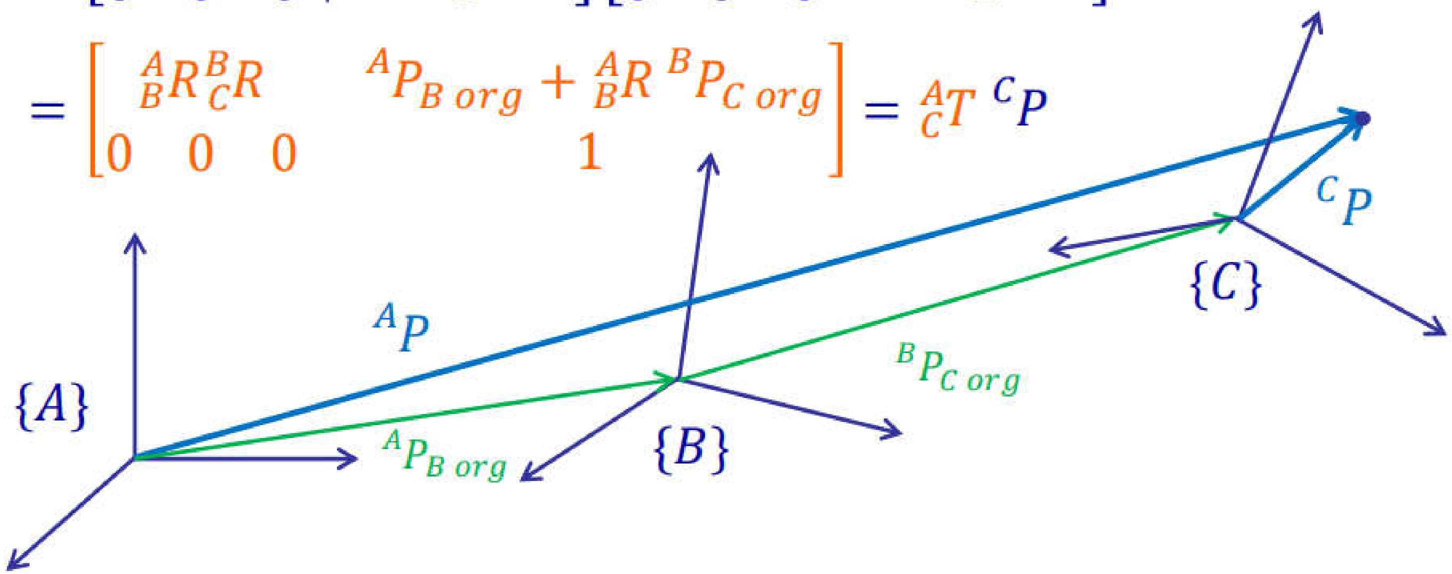

$$ {}^{A}P = {}^{A}_{B}T\,{}^{B}_{C}T\,{}^{C}_{D}T\,{}^{D}P $$

$$ = \begin{bmatrix} {}^{A}_{B}R\,{}^{B}_{C}R\,{}^{C}_{D}R & {}^{A}P_{B\,org} + {}^{A}_{B}R\,{}^{B}P_{C\,org} + {}^{A}_{B}R\,{}^{B}_{C}R\,{}^{C}P_{D\,org} \\ 0\quad 0\quad 0 & 1 \end{bmatrix} = {}^{A}_{D}T\,{}^{D}P $$

2.7 变换矩阵的运算法则

- 反矩阵 $_B^AT=\begin{bmatrix} _B^AR & ^AP_{B\,org} \\ 0\quad 0\quad 0 & 1 \end{bmatrix}$ $\qquad _A^BT={_B^AT}^{-1}=?$

2.7 变换矩阵的运算法则

- 反矩阵 $^{A}_{B}T = \begin{bmatrix} ^{A}_{B}R & ^{A}P_{B\,org} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix}$ $\qquad ^{B}_{A}T = {^{A}_{B}T}^{-1} = ?$

$^{A}_{B}T\,^{B}_{A}T = {^{A}_{B}T}\,{^{A}_{B}T}^{-1} = I_{4\times 4}$

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$$\begin{bmatrix} & ^{A}_{B}R & & ^{A}P_{B\,org} \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} & ^{B}_{A}R & & ^{B}P_{A\,org} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2.7 变换矩阵的运算法则

- 反矩阵 ${}^A_BT = \begin{bmatrix} {}^A_BR & {}^AP_{B\,org} \\ 0\quad 0\quad 0 & 1 \end{bmatrix}$ ${}^B_AT = {}^A_BT^{-1} = ?$

$${}^A_BT\,{}^B_AT = {}^A_BT\,{}^A_BT^{-1} = I_{4\times 4}$$

$$\begin{bmatrix} {}^A_BR & {}^AP_{B\,org} \\ 0\quad 0\quad 0 & 1 \end{bmatrix}\begin{bmatrix} {}^B_AR & {}^BP_{A\,org} \\ 0\quad 0\quad 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} {}^A_BR\,{}^B_AR & {}^AP_{B\,org} + {}^A_BR\,{}^BP_{A\,org} \\ 0\quad 0\quad 0 & 1 \end{bmatrix} = \begin{bmatrix} & I_{3\times 3} & & \begin{matrix}0\\0\\0\end{matrix} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2.7 变换矩阵的运算法则

- 反矩阵 $^{A}_{B}T=\begin{bmatrix} ^{A}_{B}R & ^{A}P_{B\,org} \\ 0\quad 0\quad 0 & 1 \end{bmatrix}$　　$^{B}_{A}T={}^{A}_{B}T^{-1}=?$

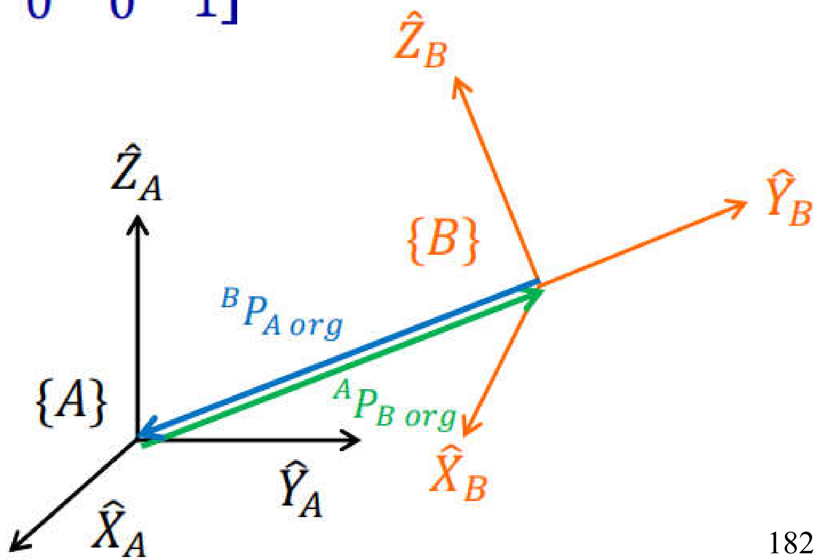
$$^{A}_{B}T\,^{B}_{A}T={}^{A}_{B}T\,^{A}_{B}T^{-1}=I_{4\times 4}$$

$$\begin{bmatrix} ^{A}_{B}R & ^{A}P_{B\,org} \\ 0\quad 0\quad 0 & 1 \end{bmatrix}\begin{bmatrix} ^{B}_{A}R & ^{B}P_{A\,org} \\ 0\quad 0\quad 0 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} ^{A}_{B}R\,^{B}_{A}R & ^{A}P_{B\,org}+{}^{A}_{B}R\,^{B}P_{A\,org} \\ 0\quad 0\quad 0 & 1 \end{bmatrix}=\begin{bmatrix} & & & 0 \\ & I_{3\times 3} & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^{A}_{B}R\,^{B}_{A}R=I_{3\times 3}$$

$$\Rightarrow {}^{B}_{A}R={}^{A}_{B}R^{T}$$



2.7 变换矩阵的运算法则

- 反矩阵 ${}^A_BT = \begin{bmatrix} {}^A_BR & {}^AP_{B\,org} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix}$　　${}^B_AT = {}^A_BT^{-1} = ?$

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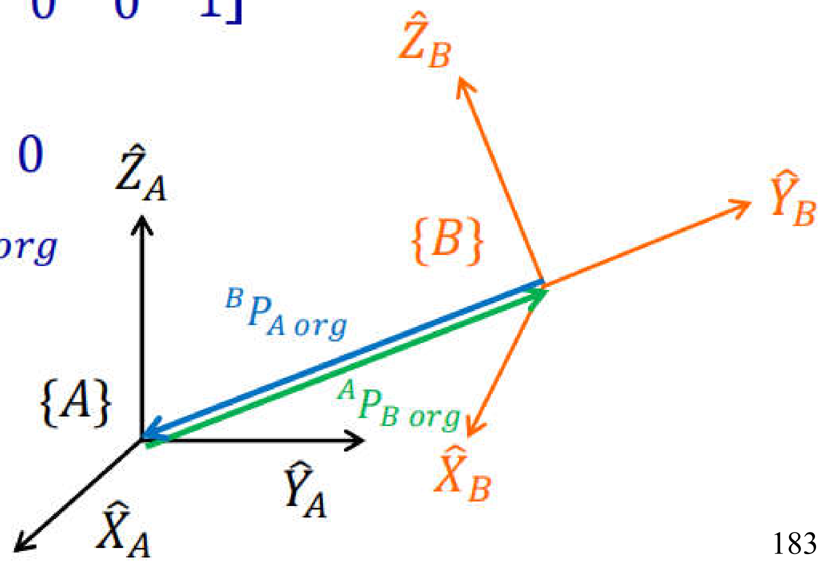
$$= \begin{bmatrix} {}^A_BR\,{}^B_AR & {}^AP_{B\,org} + {}^A_BR\,{}^BP_{A\,org} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix} = \begin{bmatrix} & & & 0 \\ & I_{3\times3} & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^A_BR\,{}^B_AR = I_{3\times3} \qquad {}^AP_{B\,org} + {}^A_BR\,{}^BP_{A\,org} = 0$$

$$\Rightarrow {}^B_AR = {}^A_BR^T \qquad \Rightarrow {}^BP_{Aorg} = -{}^A_BR^T\,{}^AP_{Borg}$$



2.7 变换矩阵的运算法则

- 反矩阵 $^A_BT = \begin{bmatrix} ^A_BR & ^AP_{B\,org} \\ 0\quad 0\quad 0 & 1 \end{bmatrix}$ $\quad ^B_AT = {}^A_BT^{-1} = ?$

$$^A_BT\,^B_AT = {}^A_BT\,^A_BT^{-1} = I_{4\times 4}$$

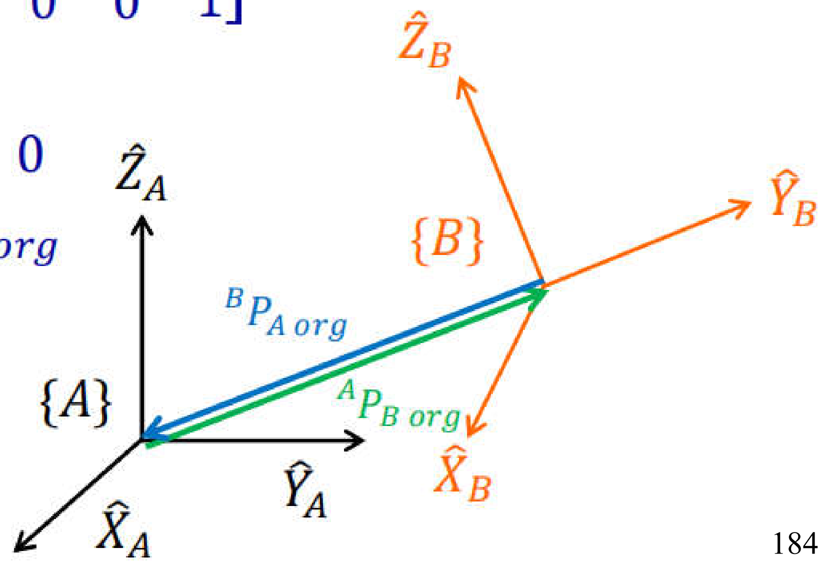
$$\begin{bmatrix} ^A_BR & ^AP_{B\,org} \\ 0\quad 0\quad 0 & 1 \end{bmatrix}\begin{bmatrix} ^B_AR & ^BP_{A\,org} \\ 0\quad 0\quad 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} ^A_BR\,^B_AR & ^AP_{B\,org} + {}^A_BR\,^BP_{A\,org} \\ 0\quad 0\quad 0 & 1 \end{bmatrix} = \begin{bmatrix} & I_{3\times 3} & & 0 \\ & & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^A_BR\,^B_AR = I_{3\times 3} \qquad ^AP_{B\,org} + {}^A_BR\,^BP_{A\,org} = 0$$

$$\Rightarrow {}^B_AR = {}^A_BR^T \qquad \Rightarrow {}^BP_{Aorg} = -{}^A_BR^T\,^AP_{Borg}$$

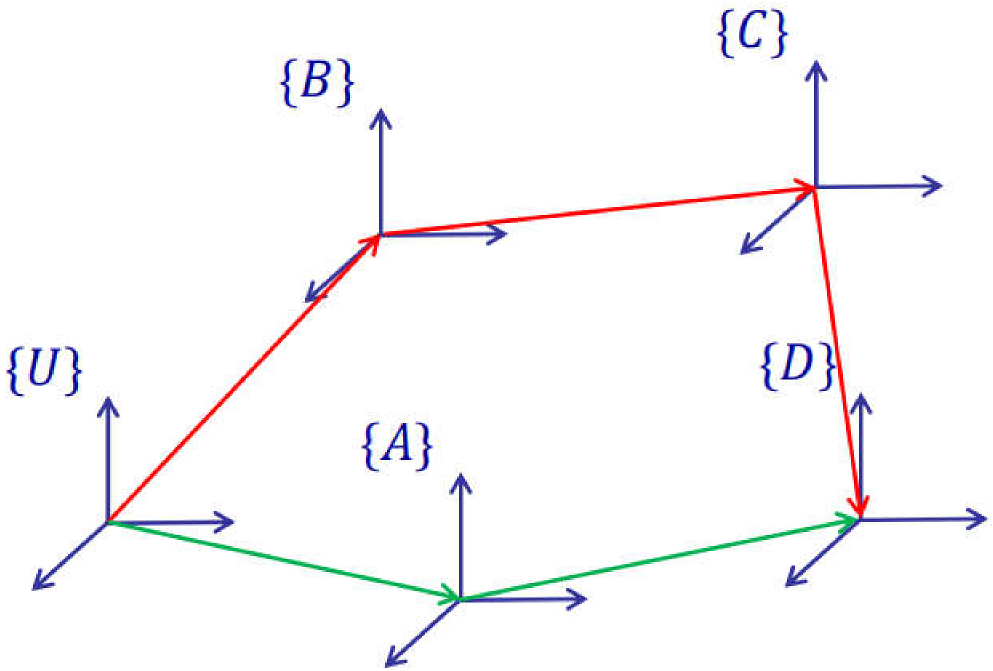
$$\Rightarrow \quad ^A_BT^{-1} = \begin{bmatrix} ^A_BR^T & -{}^A_BR^T\,^AP_{B\,org} \\ 0\quad 0\quad 0 & 1 \end{bmatrix}$$



2.7 变换矩阵的运算法则

- 連續運算，求未知之相對關係

$${}^{U}_{D}T = {}^{U}_{A}T\,{}^{A}_{D}T = {}^{U}_{B}T\,{}^{B}_{C}T\,{}^{C}_{D}T$$



2.7 变换矩阵的运算法则

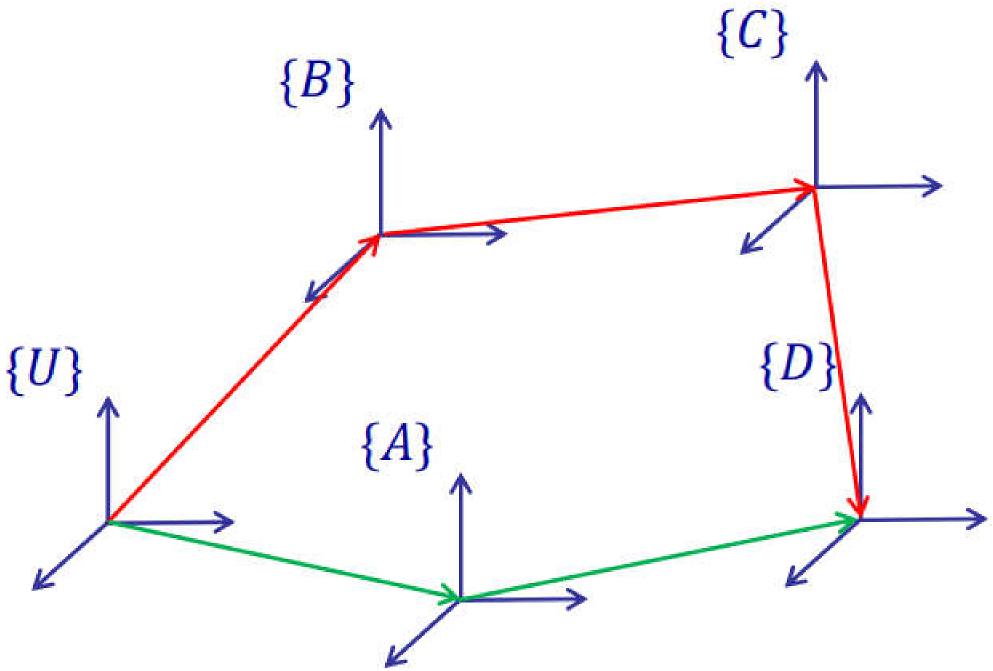
❑ 連續運算，求未知之相對關係

$$ {}^{U}_{D}T = {}^{U}_{A}T\,{}^{A}_{D}T = {}^{U}_{B}T\,{}^{B}_{C}T\,{}^{C}_{D}T $$

if ${}^{C}_{D}T$ unknown

$$ = ({}^{U}_{B}T\,{}^{B}_{C}T)^{-1}\,{}^{U}_{A}T\,{}^{A}_{D}T $$

$$ = {}^{B}_{C}T^{-1}\,{}^{U}_{B}T^{-1}\,{}^{U}_{A}T\,{}^{A}_{D}T $$



2.7 变换矩阵的运算法则

- 連續運算，求未知之相對關係

$$ {}^{U}_{D}T = {}^{U}_{A}T\,{}^{A}_{D}T = {}^{U}_{B}T\,{}^{B}_{C}T\,{}^{C}_{D}T $$

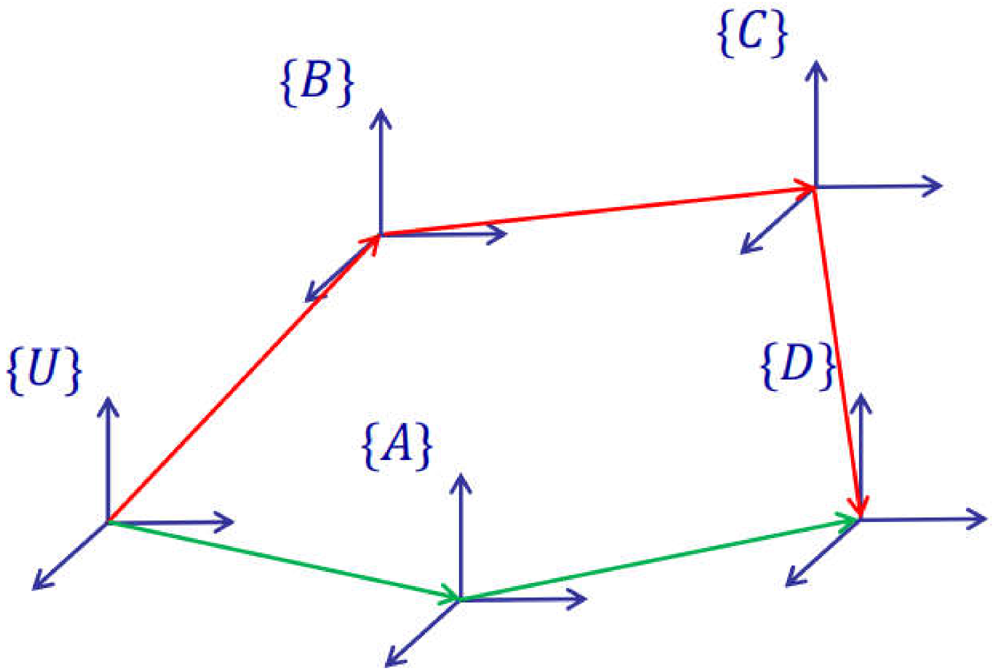
if ${}^{C}_{D}T$ unknown

$$ = ({}^{U}_{B}T\,{}^{B}_{C}T)^{-1}\,{}^{U}_{A}T\,{}^{A}_{D}T $$

$$ = {}^{B}_{C}T^{-1}\,{}^{U}_{B}T^{-1}\,{}^{U}_{A}T\,{}^{A}_{D}T $$

if ${}^{B}_{C}T$ unknown

$$ = {}^{U}_{B}T^{-1}\,{}^{U}_{A}T\,{}^{A}_{D}T\,{}^{C}_{D}T^{-1} $$



2.7 变换矩阵的运算法则

- 連續運算 法則
  - Initial condition: $\{A\}$ and $\{B\}$ coincide $\quad {}^{A}_{B}T = I_{4\times 4}$

2.7 变换矩阵的运算法则

- 連續運算 法則
  - Initial condition: $\{A\}$ and $\{B\}$ coincide $\quad {}^{A}_{B}T = I_{4\times 4}$
  - $\{B\}$ 對 $\{A\}$ 的轉軸旋轉：用 “premultiply”

2.7 变换矩阵的运算法则

- 連續運算 法則
  - Initial condition: {A} and {B} coincide $\quad {}_B^AT = I_{4\times4}$
  - $\{B\}$ 對 $\{A\}$ 的轉軸旋轉：用 "premultiply"
    - 以operator來想，對某一個向量，「以同一個座標為基準」，進行轉動或移動的操作
    - Ex: $\{B\}$依序經過T_1、T_2、T_3三次transformations

$$ {}_B^AT = T_3T_2T_1I \qquad v' = {}_B^ATv = T_3T_2T_1v $$

2.7 变换矩阵的运算法则

- 連續運算 法則
  - Initial condition: {A} and {B} coincide $\,^{A}_{B}T = I_{4\times4}$
  - $\{B\}$ 對 $\{A\}$ 的轉軸旋轉：用 “premultiply”
    - 以operator來想，對某一個向量，「以同一個座標為基準」，進行轉動或移動的操作
    - Ex: $\{B\}$依序經過T_1、T_2、T_3三次transformations

$$^{A}_{B}T = T_3T_2T_1I \qquad v' = {}^{A}_{B}Tv = T_3T_2T_1v$$

  - $\{B\}$ 對 $\{B\}$ 自身的轉軸旋轉：用“postmultiply”

2.7 变换矩阵的运算法则

連續運算 法則

- Initial condition: $\{A\}$ and $\{B\}$ coincide $\quad {}^{A}_{B}T = I_{4\times4}$
- $\{B\}$ 對 $\{A\}$ 的轉軸旋轉：用 "premultiply"
  - 以operator來想，對某一個向量，「以同一個座標為基準」，進行轉動或移動的操作
  - Ex: $\{B\}$依序經過T_1、T_2、T_3三次transformations

$$ {}^{A}_{B}T = T_3T_2T_1I \qquad v' = {}^{A}_{B}Tv = T_3T_2T_1v $$

- $\{B\}$ 對 $\{B\}$ 自身的轉軸旋轉：用"postmultiply"
  - 以mapping來想，對某一個向量，從最後一個frame「逐漸轉動或移動」來回到第一個frame
  - Ex: $\{B\}$依序經過T_1、T_2、T_3三次transformation

$$ {}^{A}_{B}T = IT_1T_2T_3 \qquad {}^{A}P = {}^{A}_{B}T\,{}^{B}P = IT_1T_2T_3\,{}^{B}P $$

2.7 变换矩阵的运算法则

- 連續運算 小結
  - 以固定的{A}或移動的{B}為基準進行移動換轉動操作，transformation matrix應用不同的連乘方式
  - 思考邏輯和考量Fixed angles vs. Euler angles的連續旋轉順序相似

谢 谢！